TOPIC 6
MATHEMATICS

6.1 Powers and logarithms

6.1.1 Powers

We use the notation of ‘powers’ as a shorthand in both *arithmetic* and *algebra*. For example, 10^3 means $10 \times 10 \times 10$ and r^2 means $r \times r$. The superscript (3 or 2 in these examples) is the ‘power’ to which the number or quantity is raised, and is called the **index** or **exponent**.

When there is a multiplication involving the same number (or quantity) raised to two different powers, the exponents (indices) are added together. For example,

$$\begin{aligned} 10^3 \times 10^2 &= (10 \times 10 \times 10) \times (10 \times 10) \\ &= 10^5 = 10^{(3+2)} \end{aligned} \tag{6.1}$$

and

$$4r^2 \times 3r^3 = 3 \times 4 \times r^2 \times r^3 = 12r^5 \tag{6.2}$$

Similarly, when there is a division the exponents are subtracted from one another:

$$10^5/10^2 = 100\,000/100 = 10^3 = 10^{(5-2)} \tag{6.3}$$

and

$$20r^3/4r^2 = 5r = 5r^{(3-2)} \tag{6.4}$$

As Equation 6.4 shows, raising any quantity to the power of ‘one’ leaves it unchanged.

If the two powers are the same, the result gives a meaning to a power of zero. For example:

$$r^3/r^3 = r^{(3-3)} = r^0 \tag{6.5a}$$

but we also can see that

$$r^3/r^3 = 1 \tag{6.5b}$$

Comparing both versions of Equation 6.5 we can see that $r^0 = 1$. Raising *any* quantity or number to the power of zero gives a result of one.

When a quantity is raised to one power and then another, the powers are multiplied together. For example:

$$(r^3)^2 = (r \times r \times r) \times (r \times r \times r) = r^6 = r^{(3 \times 2)} \tag{6.6}$$

QUESTION 6.1

Write the following expressions in as compact a form as possible.

(i) $3a^3 \times 5a^4$ (ii) $6b^5 \div 2b^2$ (iii) $3c \times 4c^2 \div 2c^3$ (iv) $(2d^2)^4$

6.1.2 Negative exponents

The pattern of division can be extended to give a meaning to **negative exponents**, for example:

$$10^3/10^5 = 10^{(3-5)} = 10^{-2} \tag{6.6a}$$

and similarly

$$10^3/10^5 = 1000/100\,000 = 1/100 = 1/10^2 \tag{6.6b}$$

Comparing both versions of Equation 6.6 we can see that raising a quantity to negative exponent must be interpreted as 'one over' the same quantity raised to the positive exponent, i.e. its **reciprocal**.

To find the reciprocal of a fraction, just turn it 'upside down'. For example,

$$\frac{1}{2/5} = \frac{1}{0.4} = 2.5 = \frac{5}{2}$$

For any quantities p and r

$$\frac{1}{p/r} = \frac{r}{p} \tag{6.7}$$

Finding the reciprocal of a reciprocal gets you back to where you started:

$$\frac{1}{5} = 0.2 \text{ and } \frac{1}{0.2} = 5$$

For any quantity r,

$$1/(1/r) = r \tag{6.8a}$$

or, using exponents

$$(r^{-1})^{-1} = r^1 = r \tag{6.8b}$$

For any quantity expressed purely as a power, the reciprocal is found by changing the sign of the exponent:

$$1/r^a = (r^a)^{-1} = r^{-a} \tag{6.9}$$

If there is another number multiplying the 'power' then you need to find its reciprocal as well. For example

$$1/(5 \times 10^3) = (5 \times 10^3)^{-1} = 5^{-1} \times 10^{-3} = 0.2 \times 10^{-3}$$

(which could also be written as $10^{-3}/5$ or as 2×10^{-4}).

Reciprocals can easily be found on a calculator using the $1/x$ button. Enter the number whose reciprocal you want to find (i.e. x) then press $1/x$.

QUESTION 6.2

Write each of the following expressions in as compact a form as possible.

(i) $1/(0.5a^4)$ (ii) $(3b^2)^{-1}$

QUESTION 6.3

Use the $1/x$ button on a calculator to evaluate each of the following reciprocals.

(i) $1/125$ (ii) 35^{-1}

6.1.3 Fractional exponents

We can give a meaning to **fractional exponents**. For example, a square with sides of length d has area A where

$$d^2 = A$$

taking the square root of each side we can write

$$d = \sqrt{A}$$

Taking the **square root** means halving the exponent of d^2 to get just d (i.e. d^1). Since we must always do the same thing to both sides of an *equation* we must also halve the exponent of A:

$$d = A^{1/2} = A^{0.5} = \sqrt{A} \tag{6.10}$$

(Notice that the power can be written either as a fraction or a decimal.)
Some other fractional exponents also have an obvious meaning. For example, $x^{-1/2}$ is the reciprocal of $\sqrt{x}$, and $x^{1.5}$ is x^2 divided by $\sqrt{x}$, as

$$x^2/x^{1/2} = x^2 \times x^{-0.5} = x^{1.5} \tag{6.11a}$$

or, which comes to exactly the same thing, it is the square root of x^3

$$(x^3)^{1/2} = x^{3/2} = x^{1.5} \tag{6.11b}$$

Fractional exponents can be combined in multiplication and division in exactly the same way as those that are whole numbers.

Square roots can be found on a calculator using the $\sqrt{x}$ button. Key in the number then press $\sqrt{x}$. A number raised to any exponent can be found using the y^x button. For example, to calculate $7^{0.37}$, enter 7, then press y^x then enter 0.37 to get the answer 2.054….

Note that some functions on your calculator may need you to press the SHIFT key first. For example on many calculators y^x is often operated by pressing SHIFT then y^x.

QUESTION 6.4

Write the following expressions in as compact a form as possible.

(i) $2a^{1.5} \times 3\sqrt{a}$ (ii) $6b/2\sqrt{b}$ (iii) $\sqrt{4c^2/d^2}$

QUESTION 6.5

Use the or y^x button on a calculator, as appropriate, to evaluate each of the following expressions.
(i) $10^{0.5}$ (ii) 17^3 (iii) $2.4^{1.7}$

6.1.4 Scientific notation

In astronomy and planetary science we sometimes need to deal with very large or very small numbers. Rather than writing out strings of zeroes we usually express such numbers in **scientific notation** (which is also known as ‘standard form’). In this notation, all numbers are expressed as some number with just one figure before the decimal point, multiplied by a power of ten.

EXAMPLE 6.1

There are said to be 1460 000 000 000 000 000 000 litres of water stored on Earth. How can this be expressed in scientific notation?

The quantity of water can be written as
$1.46 \times 1000\,000\,000\,000\,000\,000\,000$ litres $= 1.46 \times 10^{21}$ litres.

EXAMPLE 6.2

The mass of a proton is 1.67×10^{-27} kg. What is this when written out in full?

$10^{-27} = 0.000\,000\,000\,000\,000\,000\,000\,000\,001$, so the mass of a proton is $0.000\,000\,000\,000\,000\,000\,000\,000\,001\,67$ kg.

As Examples 6.1 and 6.2 illustrate, scientific notation provides a compact way of writing very large or small numbers. It enables you to compare the sizes of two quantities at a glance; rather than having to count all the zeroes, you just have to look at the powers of ten.

Numbers in scientific notation can be entered into a calculator using the button marked E, EE or EXP. This button can be read as ‘times ten to the power of …’. For example, to enter 1.46×10^{21} you key in 1.46, next press the E button and then key in 21. The calculator will probably display something like 1.46 ^{21} or 1.46 E 21 — most displays do not show the 10.

Beware of two common pitfalls. First, if you key in 1.46 × 10 × E 21 you have actually entered a number that is ten times too big because the calculator interprets it as $1.45 \times 10 \times 10^{21} = 1.46 \times 10^{22}$. Second, take care how you copy down numbers from a calculator display. If you write down the displayed number as 1.46^{21} you have actually written 1.46 × 1.46 × … (21 times altogether) which comes to a bit over 2827.5 and is certainly not the same as 1.46×10^{21}.

To enter a power of ten with a negative exponent, you need to use the +/– button directly before or after entering the exponent. For example, 1.67×10^{-27} can be entered by keying in 1.67 E +/– 27 or 1.67 E 27 +/–. The +/– button changes the sign of a number; pressing it twice gets you back to the original sign.

Avoid the temptation to use the arithmetic minus button when entering negative exponents, as the calculator will interpret this as subtracting one number from the previous one. For example, if you type in 1.67 E – 27 a calculator will interpret this as $1.67 \times 10^{0} - 27$, i.e. $1.67 - 27$ which comes to -25.33 and is not what was intended.

QUESTION 6.6

Write the following quantities in scientific notation.

(i) 150 000 000 000 m (the mean Earth–Sun distance)

(ii) 0.000 000 000 000 000 000 160 C (the charge of a proton).

QUESTION 6.7

Use a calculator to evaluate each of the following expressions.

(i) $4.57 \times 10^{14} \times 6.63 \times 10^{-34}$

(ii) $3.00 \times 10^{8}/4.86 \times 10^{-7}$.

6.1.5 Logarithms

The numbers 1, 10, 100, 1000 etc, and 0.1 0.01, 0.001 etc, can all be expressed as whole number powers of ten: 10^0, 10^1, 10^2, 10^3 etc. and 10^{-1}, 10^{-2}, 10^{-3} etc. Fractional powers of ten produce other numbers: for example, $10^{0.5} = 3.162\ldots$. *Any* positive number can be expressed as a power of ten, and the powers can in principle be found by reading from the *graph* in Figure 6.1: if you want to express some number y as a power of ten, then read the corresponding value of x from the graph.

The power to which ten must be raised to produce a given value of y is called the **logarithm** to base ten, or base-ten logarithm, of y, also known as the *common logarithm* of y or, often, just 'the logarithm' or 'the log' of y. In symbols,

$$y = 10^x \quad \text{or} \quad x = \log_{10}(y) \tag{6.12}$$

The symbols $\log(y)$ and $\lg(y)$ are also used in place of $\log_{10}(y)$. (In principle any number can be used as the base for a system of logarithms e.g. if $y = 2^x$, $x = \log_2(y)$. However, the only number apart from 10 that is commonly used for logs is the number e = 2.718… which has special mathematical properties. However, logs to base e, or so-called *natural logarithms* do not feature in S282 or S283.)

The logarithms of whole-number powers of ten can be written down quite easily. For example, 1000 can be written as 10^3, so $\log(1000) = 3$. Likewise, 0.01 can be written as 10^{-2}, so $\log(0.01) = -2$.

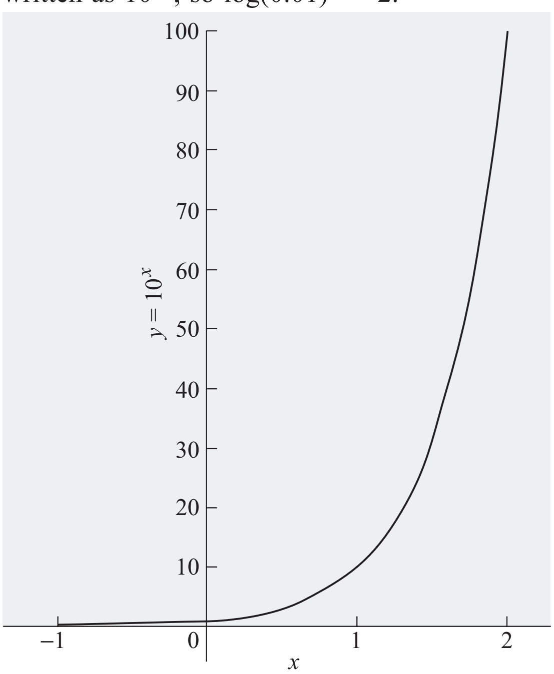

Figure 6.1 A graph of $y = 10^x$ plotted against x.

The logarithm of any positive number can be found using the LOG button on a calculator. Key in the number then press log. Note that negative numbers do not have logarithms, since raising ten to any positive or negative power produces a positive result. Note too that all numbers greater than 1 have positive logarithms while numbers between one and zero have negative logarithms. Since $10^0 = 1$, $\log_{10}(1) = 0$.

QUESTION 6.8

Without using a calculator, write down the logarithms to base ten of the following numbers.

(i) 10 000 (ii) 0.000 01 (iii) 1

QUESTION 6.9

Use a calculator to find the base-ten logarithms of the following numbers.

(i) 2 (ii) 3.172 277 (iii) 0.625

QUESTION 6.10

Find the value of x for each of the following values of 10^x.

(i) 2 (ii) 3.172 277 (iii) 0.625

The process of finding the common logarithm of a number can be reversed. As Equation 6.12 shows, if we take the base-ten log of some number y to get another number x, then finding 10^x gets us back to our original number y. The number y is the base-ten antilogarithm or **antilog** of x:

$$x = \log_{10}(y),\ y = \text{antilog}_{10}(x) \tag{6.13}$$

The base-ten antilogs of whole numbers can be found quite easily by working out the appropriate power of ten. For example,

$$\text{antilog}_{10}(3) = 10^3 = 1000$$

$$\text{antilog}_{10}(-2) = 10^{-2} = 0.01$$

$$\text{antilog}_{10}(0) = 10^0 = 1$$

For other numbers, antilogs can be found on a calculator either by using the y^x button to raise ten to the power of that number, or by keying in the number then pressing the INV (or SHIFT) button followed by the LOG button. For example, to find the antilog of 1.2, either key in 1.2 then press INV and LOG to get the answer 15.848…, or key in 10, press y^x and key in 1.2 to get 15.848….

QUESTION 6.11

By calculating 10^x with the y^x button and/or by using the INV and LOG buttons use a calculator to find $\text{antilog}_{10}(x)$ for each of the following values of x.

(i) 0.5 (ii) 1.23 (iii) −1.23

6.2 Precision

6.2.1 Significant figures

The number of **significant figures** in a quantity is, simply, the number of figures that give meaningful information about its size and precision. For example, if you measure the length of a table to the nearest cm (0.01 m) and find that it was 1.72 m, then the quantity 1.72 m has three significant figures. All three figures carry some meaning about the size of the quantity, and the final '2' indicates that the length really is 1.72 m and not 1.71 m or 1.73 m.

The length written as 1.72 m has two **decimal places** (two figures after the decimal point). The number of decimal places changes according to how a quantity is written but the number of significant figures does not. You could for example write the length 1.72 m as 172 cm (no decimal places, 3 significant figures) or as 0.00172 km (five decimal places, three significant figures).

Zeroes at the beginning of a number are *never* significant. The quantity 001.72 m is exactly the same as 1.72 m and 0.00172 km. After the decimal point, zeroes are important because they give the number its 'place' but they are not significant in any other way.

Zeroes at the end of a number are a bit more tricky. After a decimal point they *are* significant. For example if you measure the distance to the corner shop to the nearest 10 m (0.01 km) and find it is 1.40 km rather than 1.39 km or 1.41 km then the zero is conveying some useful information. If you wrote that the distance was 1.4 km that would mean that the length was nearer to 1.4 km than 1.3 km or 1.5 km — in other words, that you had only measured the length to the nearest hundred metres (0.1 km). But if for some reason you wanted to express the length in metres, you might write 1400 m. The second zero is important because it gives the number its 'place', but it implies that you have measured the distance to the nearest metre and you know that it is 1400 m not 1399 m or 1401 m, so it is misleading and is certainly not significant. To avoid giving the impression that a quantity has been measured more precisely than it actually has, it is best to use *scientific notation* and write the length as 1.40×10^3 m then there is no ambiguity.

QUESTION 6.12

How many significant figures are there in each of the following numbers?

(i) 1.6 (ii) 0.016 (iii) 001.6 (iv) 1.60 (v) 1600

6.2.2 Significant figures in calculations

When two or more quantities are multiplied or divided by one another, the final result cannot be any more precisely-known than the *least* precise quantity used in the calculation. This means that the answer can only have as many significant figures as the least precise quantity. So in any calculation you should always round the *final* answer to a suitable number of significant figures and avoid the temptation to write down the whole calculator display. The following example illustrates this in detail, and the rest of the examples in this booklet, and our answers to the questions, provide many further examples of correct numbers of 'sig figs'.

EXAMPLE 6.3

Suppose you measure the length and width of a table to be 1.72 m and 0.94 m. What is its area?

On a calculator,

$$\text{area} = \text{length} \times \text{width} = 1.72\,\text{m} \times 0.94\,\text{m} = 1.6168\,\text{m}^2$$

But the length and width have only been measured to the nearest 0.01 m so they could be as much as 0.005 m away in either direction. They could be large as 1.725 m and 0.945 m, which would give a calculator display of 1.620 125. Or they could be as small as 1.715 m and 0.935 m which would give a calculator display of 1.603 525. Most of the figures after the decimal point are meaningless — they are not significant.

The least precise value used in the calculation is the width, 0.94 m, which has two significant figures. Rounding any of the calculator displays to two figures produces an answer 1.6 m^2, which makes sense because it is compatible with the possible range of lengths and widths and does not imply that the area can be known any more precisely.

Whole numbers arising from the mathematics of a situation rather than from physical measurement really are exact whole numbers. For example, the *circumference* of a circle with radius r is $2\pi r$ and the ‘2’ is exactly 2, even if you measure it to a huge (infinite) number of decimal places. So it does not limit the precision to which the circumference can be calculated. Similarly, when a number is written in scientific notation the ‘10’ raised to a power is exactly 10 and does not reduce the precision of the overall number. For example, 1.40×10^3 m has three significant figures.

EXAMPLE 6.4

A meteorite of mass 1.65 kg travels at a speed of 2.3 km s^{-1}. What is its *kinetic energy* E_k? ($E_k = \frac{1}{2}mv^2$) How many figures in the answer are significant?
On a calculator,

$$E_k = \frac{1.65\,\text{kg} \times (2.3 \times 10^3\ \text{m}\,\text{s}^{-1})^2}{2} = 4364\,250\,\text{J or } 4.364\,25 \times 10^6\ \text{J}$$

As the speed is only known with a precision of 2 significant figures, only two figures of the answer are significant so it should be rounded to 4.4×10^6 J.

If the figure immediately following the final significant one is five or greater, as in Example 6.4, then the preceding value should be rounded up. Sometimes this has a knock-on effect. For example suppose you calculate an answer as 3.987 and you know that only two figures are significant. The second significant figure here is the ‘9’ after the decimal point, and the one following it is ‘8’ so the ‘9’ must be rounded up which in turn means that the ‘3; becomes a ‘4. The final result should be written 4.0 not just 4, because the zero after the point *is* significant.

QUESTION 6.13

A space probe (e.g. Apollo 12) takes 33 hours to travel a distance of 241×10^6 m from the Earth to the Moon. What is its average speed? How many figures in the answer are significant? (Speed = distance/time, 1 hour = 3600 s.)

6.2.3 Orders of magnitude

Sometimes when doing scientific calculations we are not interested in the precise answer but just need to know its **order of magnitude** — in other words, the nearest power of ten. For example, the proton's charge is, $e = 1.60 \times 10^{-19}$ C to three *significant figures* but when expressed as an order of magnitude is simply 10^{-19} C; we can say that it is 'of the order of 10^{-19} C' (or 'of order 10^{-19} C'). In symbols

$$e \sim 10^{-19}\ \mathrm{C}$$

The radius, r, of the Sun is 6.96×10^7 m, and to the nearest order of magnitude this is 10^8 m.

$$r \sim 10^8\ \mathrm{m}$$

Notice that the value for the proton charge is rounded down while that of the Sun's radius is rounded up.

QUESTION 6.14

Express the following quantities to the nearest order of magnitude.

(i) Boltzmann constant, $k = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$

(ii) The number 4π

(iii) The Sun's mass, 1.99×10^{30} kg

(iv) The distance of Jupiter from the Sun, 7.78×10^{11} m

QUESTION 6.15

In a vacuum, light travels at $3.00 \times 10^8\ \mathrm{m\,s^{-1}}$. There are 3.1536×10^7 seconds in one year. To the nearest order of magnitude how far does light travel through space in one year? (Use distance = speed × time, and do not use a calculator!)

6.3 Algebra and equations

6.3.1 Equations

An **equation** is a mathematical statement that two quantities are *equal*, that is, exactly the same as one another. An equation involves two different combinations of numbers and quantities, linked by an equals sign, =. Generally we use equations to relate two or more physical quantities to one another, and the quantities are represented by letters as this makes it easier to manipulate the equations. The general process of using and manipulating equations involving symbols is called **algebra**.

For example, *speed* is conventionally represented by u, v or (particularly for electromagnetic radiation) by c, distance by x, s or d, and time interval by t. These three quantities are related by an equation:

$$\text{distance travelled} = \text{speed} \times \text{time interval} \tag{6.14a}$$

or

$$x = ut \tag{6.14b}$$

In Equation 6.14, distance, x, is the **subject** of the equation; the equation shows how it can be calculated by substituting values for the quantities on the right-hand side.

Any equation can be rearranged to make another quantity the subject. In rearranging an equation, the crucial thing to remember is that both sides must always be equal. To maintain this situation, you must always do *exactly* the same thing to both sides of an equation.

When rearranging an equation, it can sometimes be helpful if you start off swapping the left- and right-hand sides of an equation so that the part with the quantity you are interested in is on the left-hand side. (You don't *have* to do this, but it can make it easier to see where you are going. And if you *do* this, be careful *only* to exchange left and right sides — don't change anything else!) Next you need to 'undo' all the things that involve that quantity and hence get it on its own. If the quantity is multiplied by something, then you can 'undo' that by division. For example, to make speed, u, the subject of Equation 6.14, you can write

$$ut = x$$

then divide both sides by t:

$$ut/t = x/t$$

On the left-hand side ut/t is just the same as u (the two 't's cancel) so we have

$$u = x/t \tag{6.15}$$

You can undo division by multiplication and similarly you can undo addition by subtraction and vice versa.

EXAMPLE 6.6

An object's acceleration, a, can be calculated from its initial speed, u, final speed, v, and the time interval, t, taken for the speed to change from u to v:

$$a = (v - u)/t \tag{6.16}$$

Given the initial speed, acceleration, and time interval, how can you calculate the final speed? To make v the subject, first write

EXAMPLE 6.5

An object's acceleration, a, can be calculated from its initial speed, u, final speed, v, and the time interval, t, taken for the speed to change from u to v:

$$a = (v - u)/t \tag{6.16}$$

Given the initial speed, acceleration, and time interval, how can you calculate the final speed? To make v the subject, first write

$$(v - u)/t = a$$

Then multiply both sides by t:

$$t(v - u)/t = at$$

so $(v - u) = at$

and add u to both sides:

$$u + v - u = u + at$$

thus $v = u + at$ (6.17)

Notice the order in which we do and undo the various steps. In Equation 6.16 we would first subtract u from v then divide by t. We have to undo the steps in the reverse order, so we undid the division first, because the whole of $(v - u)$ was divided by t, and then we were able to undo the subtraction of u from v.

To undo a square, take a *square root* and to undo a *reciprocal* take another reciprocal. The key thing is always to think of *doing the same thing to both sides* as the following examples show.

EXAMPLE 6.6

The *kinetic energy* E_k of an object mass m moving at speed v is given by

$$E_k = \frac{mv^2}{2} \tag{6.18}$$

How can you calculate speed from given values of E_k and m?

You need to make v the subject of the equation. Write

$$\frac{mv^2}{2} = E_k$$

To get v on its own, first get v^2 on its own: multiply both sides by 2 and divide by m:

$$\frac{2mv^2}{2m} = \frac{2E_k}{m}$$

so $v^2 = \dfrac{2E_k}{m}$

Then take the square root of both sides:

$$\sqrt{v^2} = \sqrt{\frac{2E_k}{m}}$$

therefore $v = \sqrt{\dfrac{2E_{\mathrm{k}}}{m}}$ (6.19)

Notice that the square root covers *everything* on the right-hand side.

EXAMPLE 6.7

The *density* ρ of a substance is related to the mass m and volume V of a sample.

$$\rho = \frac{m}{V} \tag{6.20}$$

How can you calculate the volume occupied by a given mass of a substance if you know its density?

To make V the subject of the equation, write

$$\frac{m}{V} = \rho$$

To get V on top, take the *reciprocal* of both sides:

$$\frac{1}{(m/V)} = \frac{1}{\rho}$$

so $\dfrac{V}{m} = \dfrac{1}{\rho}$

then multiply both sides by m:

$$\frac{mV}{m} = \frac{m}{\rho}$$

thus

$$V = \frac{m}{\rho} \tag{6.21}$$

Alternatively, you might prefer a slightly different, but equivalent, route. To start, multiply both sides of Equation 6.20 by V

$$\frac{Vm}{V} = V\rho$$

thus $m = V\rho$

Then divide both sides by ρ:

$$\frac{m}{\rho} = V$$

which is exactly the same as Equation 6.21 when you interchange the left- and right-hand sides.

QUESTION 6.16

Force F, mass m and acceleration a are related by the Equation $F = ma$. Rearrange this equation to make a the subject.

QUESTION 6.17

The *gravitational force*, F, between two masses M and m with their centres separated by a distance r is given by the equation $F = GMm/r^2$ where G is the gravitational constant. Rearrange this equation to make r the subject.

6.3.2 Combining equations

It is often useful to combine two equations to produce a third that allows you to calculate a particular quantity. Sometimes this can allow you to tackle a problem that at first seems impossible, as the next example shows.

EXAMPLE 6.8

An apple falls from a tree through a height Δh. As it falls it loses *gravitational potential energy*

$$\Delta E_g = mg\,\Delta h$$

and gains kinetic energy

$$\Delta E_k = \frac{mv^2}{2}$$

Without knowing the mass of the apple, use these equations to calculate its speed v after it has fallen through a given height.

The key thing here is to note that the loss in gravitational energy is equal to the gain in kinetic energy. You could guess a mass, work out the loss in E_g then set that equal to the gain in E_k, then use that to find the speed, which would involve dividing by the mass that you had guessed…. It is much neater to do some algebra first and write

$$\frac{mv^2}{2} = mg\,\Delta h$$

Dividing both sides by m we have

$$\frac{v^2}{2} = g\,\Delta h$$

and so we can multiply both sides by 2 and take the *square root* to make v the subject:

$$v = \sqrt{2g\,\Delta h}$$

Sometimes the **elimination** of the unwanted quantity is even less obvious. For example, we might want to calculate the photon energy of some *electromagnetic radiation* whose *wavelength* λ we know. The *photon energy*, E_{ph}, is related to the *frequency*, f, of the radiation

$$E_{ph} = hf \qquad (6.22)$$

where h is the Planck constant. The frequency is in turn related to the wavelength:

$$c = f\lambda \qquad (6.23)$$

where c is the wave speed of electromagnetic radiation. You can of course calculate f using Equation 6.23 then put that value into Equation 6.22 to find the photon energy, but it is much neater to use algebra to produce an equation for the quantity you want and thus avoid any unnecessary calculation. To eliminate the unwanted quantity, first make f the subject of Equation 6.23:

$$f = c/\lambda$$

then because c/λ is exactly the same as f, you can put c/λ instead of f in Equation 6.22

$$E_{ph} = hc/\lambda \tag{6.24}$$

This process of replacing one symbol by another combination is called **substitution**. Notice the steps involved. First make the unwanted quantity the *subject* of an equation involving things that you know. Then use that to replace the unwanted quantity in the other equation.

An alternative way to eliminate the unwanted quantity is to make it the subject of *both* equations:

$$f = c/\lambda$$

and, from Equation 6.22

$$f = E_{ph}/h$$

Then because the right-hand sides are both equal to the unwanted quantity, they must also be equal to each other. They can therefore be linked by an equals sign to produce a new equation without the unwanted quantity:

$$E_{ph}/h = c/\lambda$$

This can be rearranged to get the desired subject as before:

$$E_{ph} = hc/\lambda$$

These methods are essentially the same, but substitution is usually less long-winded.

QUESTION 6.18

The distance around the circumference of a circle radius r is $2\pi r$. Use $v = x/t$ to derive an expression for the speed, v, of a planet that takes a time T to travel once around a circular orbit of radius r.

QUESTION 6.19

When a mass m of a substance is heated through a temperature rise ΔT, the change in its *internal energy*, Δq, is given by $\Delta q = mc\,\Delta T$ where c is the s*pecific heat capacity* of the substance. The *density* ρ of a substance is related to the mass m and volume V of a sample: $\rho = m/V$. By substituting a suitable expression for m, produce an equation that shows how to calculate Δq when a volume V of a substance with a given ρ and c is heated through ΔT.

6.4 Proportionality

6.4.1 Direct proportionality

If one quantity is **directly proportional** (or just **proportional**) to another, whenever one is multiplied or divided by a given factor then so is the other. For example, for travel at a constant speed, distance covered is proportional to the time taken: if you travel for three times as long then you cover three times the distance. Similarly, the *acceleration* of a given object is proportional to the net *force* acting on it: if you halve the force you also halve the acceleration. One quantity is related to the other by a constant factor, called a **constant of proportionality**. If some quantity y is proportional to another quantity x we can write this in various ways:

$$y = kx \tag{6.25a}$$

or

$$y/x = k \tag{6.25b}$$

where k is the constant of proportionality. Plotting a *graph* of y against x produces a straight line through the origin, with *gradient* k. Another way to write Equation 6.25 is

$$y \propto x \tag{6.25c}$$

where the sign $\propto$ is read 'is (directly) proportional to'.

In our example of travel at constant speed, u, it is the speed itself that is the constant of proportionality because distance x is related to time t via the equation

$$x = ut$$

Some equations relating physical quantities include several proportionality relationships. For example, when a mass m of a substance is heated through a temperature rise ΔT, the change in its *internal energy*, Δq, is given by

$$\Delta q = mc\,\Delta T \tag{6.26}$$

where c is the s*pecific heat capacity* of the substance. From Equation 6.26 we can identify two proportional relationships that describe how Δq behaves for a given substance (i.e. a given value of c):

$$\Delta q \propto m$$

that is if you multiply or divide the mass by a given factor then you multiply or divide Δq by the same factor. Also

$$\Delta q \propto \Delta T$$

Proportional relationships can also involve quantities with *exponents*. For example, the kinetic energy E_k of an object mass m moving at speed v is given by

$$E_k = \frac{mv^2}{2} \tag{6.27}$$

so we can write

$$E_k \propto v^2$$

If we multiply the speed v by 2, then v^2 is multiplied by 4 so E_k is also multiplied by 4 ($= 2^2$), and if we multiply the speed by 3 then v is multiplied by 9 ($= 3^2$) and so

also is E_k. Whatever factor we use to multiply or divide v, then E_k is multiplied or divided by the square of that factor.

Note that a proportional relationship always tells us how one quantity will *change* as another *changes*, so both sides of the relationship must be things that can vary. It is meaningless to write that a quantity is proportional to a constant, because a constant, by definition, does not change. So for example we can say that *rest energy* E_0 is proportional to mass m from the equation $E_0 = mc^2$, but we *cannot* write that $E_0 \propto c^2$ because c, the speed of electromagnetic radiation in a vacuum, is fixed.

QUESTION 6.20

The pressure p exerted by n molecules of gas at absolute temperature T is given by $p = nkT$ where k is the Boltzmann constant. Write proportional relationships to show how p depends on other quantities in the equation.

6.4.2 Inverse proportionality

Sometimes multiplying one quantity by a given factor results in another being divided by the same factor. For example, if you double the *frequency* of waves that are travelling at a fixed speed, then you halve their *wavelength*. And if you multiply your *speed* by three, then you cover a fixed distance in one-third of the time. In these example one quantity is **inversely proportional** to the other. (Note the correct term: there is no such thing as 'indirectly proportional'.) If some quantity y is inversely proportional to some quantity x then we can write

$$y = k/x \qquad (6.28a)$$

or

$$yx = k \qquad (6.28b)$$

where k is a constant of proportionality.

There is no separate sign meaning 'is inversely proportional to' so we write

$$y \propto 1/x \qquad (6.28c)$$

i.e. y is directly proportional to the reciprocal of x, usually read as 'y is inversely proportional to x'. As with direct proportionality, inverse proportionality can involve quantities with *exponents*.

QUESTION 6.21

Write proportionality relationships between x and y in each of the following examples.

(i) $y = 27A/x$ where A is a constant

(ii) $y = 4\pi x^2$

(iii) $y = GMm/x^2$

6.4.3 Combining proportional relationships

Proportional relationships can be combined. For example, when Newton developed his *law of gravitation* he stated that the magnitude of the gravitational *force*, F_g, between two masses m_1 and m_2, with their centres separated by a distance r, was directly proportional to each of the masses and inversely proportional to the square of their separation:

$$F_g \propto m_1$$

$$F_g \propto m_2$$

$$F_g \propto \frac{1}{r^2}$$

We can combine these expressions by multiplying all the right-hand sides together:

$$F_g \propto \frac{m_1 m_2}{r^2}$$

Here the *constant of proportionality* is G, the gravitational constant.

$$F_g = \frac{G m_1 m_2}{r^2}$$

EXAMPLE 6.9

The lifetime, t, of a star similar to our Sun is proportional to its mass M and inversely proportional to its output *power*, or luminosity, L:

$$t \propto M/L$$

Its luminosity in turn depends on its mass:

$$L \propto M^5$$

Combine these proportional relationships into one that describes how a Sun-like star's lifetime depends on its mass.

It is possible to go round in circles when dealing with situations like this, as it looks as though everything is proportional to everything else. Things become more manageable if we turn the proportionalities into equations, and for example write

$$t = kM/L$$

$$L = KM^5$$

where k and K are some *constants of proportionality* whose actual values we don't really care about. Now we can substitute for L in our first equation

$$t = \frac{kM}{KM^5}$$

As $M/M^5 = M^{-4}$ we can write

$$t = \frac{k}{K} M^{-4}$$

but as we are only interested in how t varies with M we can drop the K and k and write another proportional relationship:

$$t \propto M^{-4}$$

QUESTION 6.22

A star's output *power*, known as its luminosity, L, depends on its surface *temperature* T and on its surface area A. The surface area A in turn depends on the star's radius r.

$L \propto T^4$, $L \propto A$ and $A \propto r^2$

Combine these proportional relationships to get a single proportional relationship that described how a star's luminosity L depends on T and r.

6.4.4 Using proportionality in problems

Situations sometimes arise when we need to use a proportional relationship in a problem. For example, stars similar to our Sun have lifetimes t that depend on their mass, M, such that $t \propto M^{-4}$. The Sun's lifetime is estimated to be 10 Ga. We can ask what would be the lifetime of a similar star whose mass is twice that of the Sun.

There are several ways to tackle this. All of them boil down to the same thing but some are neater than others. In all of them, start by writing down an equation:

$$t = KM^{-4} \tag{6.29}$$

where K is a *constant of proportionality* whose value we don't know.

A long-winded way to tackle the problem would be to make K the *subject* of Equation 6.29,

$$K = t/M^{-4} = tM^4 \tag{6.30}$$

and put in values for the Sun's mass and lifetime to calculate K. Then use this same K in Equation 6.29, putting in a mass that is twice that of the Sun.

A much neater way is to use symbols t_1 and M_1 to represent the Sun's lifetime and mass, and t_2 and M_2 to represent those of the other star. We can then write:

$$t_1 = KM_1^{-4} \tag{6.31a}$$

and $$t_2 = KM_2^{-4} \tag{6.31b}$$

Rearranging gives

$$K = t_1 M_1^4$$

and $$K = t_2 M_2^4$$

As both right-hand sides are equal to K we can write

$$t_1 M_1^4 = t_2 M_2^4 \tag{6.32}$$

Dividing both sides by M_2^4 we get

$$t_2 = t_1 \left(\frac{M_1}{M_2} \right)^4 \tag{6.33}$$

and because $M_2 = 2M_1$, we can write

$$t_2 = t_1 \left(\frac{M_1}{2M_1} \right)^4 = \frac{t_1}{2^4} \tag{6.34}$$

so we can solve the problem without having to calculate K and without having to know the actual masses of the Sun and the other star.

There is a variation on this method, which gets to the same result using different steps. Starting with the two versions of Equation 6.31, we can divide the left-hand side of Equation 6.31b by the left-hand side of Equation 6.31a to get t_2/t_1. Since we are dealing with *equations*, we must do exactly the same to both sides and so we must also divide the right-hand side of Equation 6.31b by t_1. But since $t_1 = KM_1^{-4}$, we can divide the right-hand side by KM_1^{-4} to get

$$\frac{t_2}{t_1} = \frac{KM_2^{-4}}{KM_1^{-4}} = \left(\frac{M_2}{M_1}\right)^{-4} = \left(\frac{M_1}{M_2}\right)^{4}$$

which can be rearranged to get Equation 6.34 as before.

QUESTION 6.23

The orbits of planets around the Sun are described by the relationship $t^2 \propto r^3$ where t is the time to travel once around the orbit and r is the radius of the orbit. The Earth takes one year to travel once around its orbit. The planet Jupiter has an orbit whose radius is 5.2 times that of the Earth's. How long does it take Jupiter to travel once around its orbit?

6.5 Circles, angles and trigonometry

6.5.1 Circles and spheres

Circles and spheres are important shapes in astronomy and planetary science. Planets in the Solar System move in approximately circular orbits, as do many stars in their orbits in the Galaxy, and stars and planets themselves are approximately spherical.

The diameter, D, of a circle is twice its radius, r, (Figure 6.2) and the **circumference**, C, (the distance around the edge) is related to the radius and diameter

$$C = \pi D = 2\pi r \tag{6.35}$$

the number represented by π (Greek letter pi) is equal to 3.141 592 654 (to 9 *decimal places*) and is generally stored on a calculator — look for the π button.

The area, A, of a circle is related to its radius:

$$A_{\text{circ}} = \pi r^2 \tag{6.36}$$

Putting $D = 2r$ in Equation 6.36 we can also see how the area is related to the diameter:

$$A_{\text{circ}} = \pi(D/2)^2 = \pi D^2/4 \tag{6.37}$$

The surface area, A, of a sphere is also related to its radius:

$$A_{\text{sph}} = 4\pi r^2 \tag{6.38}$$

and so is its volume, V:

$$V_{\text{sph}} = 4\pi r^3/3 \tag{6.39}$$

Notice that the areas of a circle and a sphere have dimensions of length × length (that is r^2) and so they have SI units of m^2, and the volume of a sphere has dimensions of length^3 and SI units m^3; the other numbers in the equations are dimensionless.

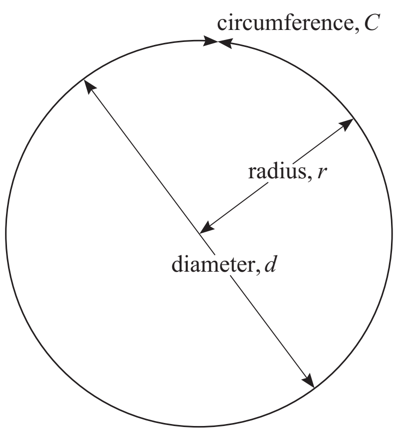

Figure 6.2 A circle.

QUESTION 6.24

Derive equations for (i) the surface area and (ii) the volume of a sphere in terms of its diameter.

QUESTION 6.25

The Earth moves around the Sun in an approximately circular orbit of radius 1.50×10^{11} m. What is the distance around the orbit?

QUESTION 6.26

The Sun is a sphere with radius 6.96×10^{8} m. What are (i) its surface area and (ii) its volume?

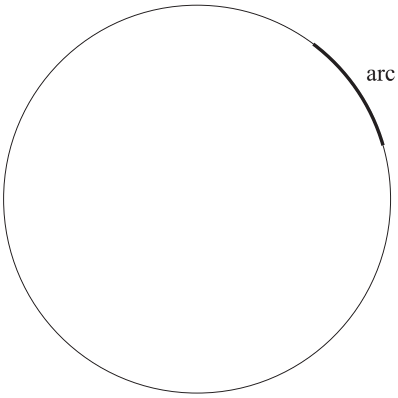

Figure 6.3 An arc of a circle.

6.5.2 Angles

In astronomy and planetary science we are often concerned with the angle between two different directions, for example the directions to either 'edge' of a planet or a nebula. Angles are often measured in **degrees of arc** (often just called **degrees**). The term **arc** refers to a part of the curved edge of a circle (Figure 6.3). There are 360 degrees of arc (360 deg or 360°) in one complete turn. A quarter-turn is 90°, and is called a **right angle**. Figure 6.4 shows some angles measured in degrees. Notice the way they are marked and labelled.

For expressing small angles, the degree of arc is divided into 60 **minutes of arc** (60 **arcmin**, 60′) and a minute of arc is further divided in 60 **seconds of arc** (60 **arcsec**, 60″). Angles in astronomy tend to be very small so the arcsec is a common unit. You will probably not often need to convert between the two ways of expressing angles measured in degrees, but these two examples show how it is done.

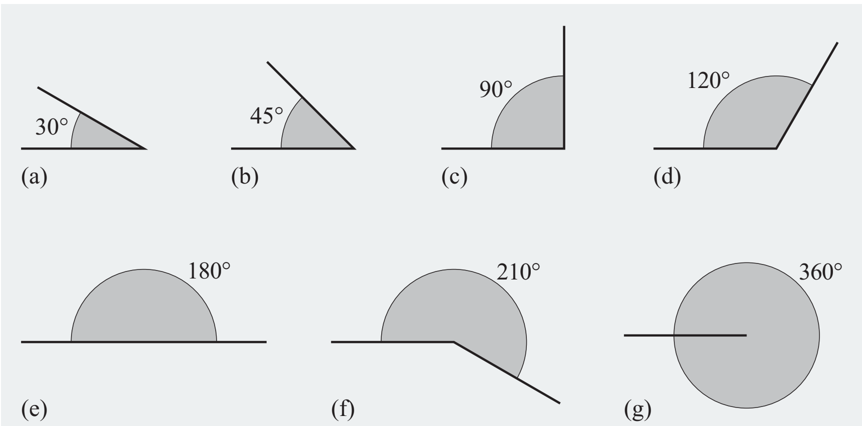

Figure 6.4 Some angles measured in degrees of arc.

EXAMPLE 6.10

A certain angle is written as 2° 24′ 30″. Express this angle as a decimal number of degrees.

Start from the small end: 30″ = 0.50′ so the angle is 2° 24.50′.

Now 24.50′ = (24.50/60) degrees = 0.4083° so the measured angle is 2.4083°.

EXAMPLE 6.11

A certain angle is written as 3.4567°. Express this in degrees, minutes and seconds of arc.

Start from the large end: the angle is 3 degrees and some number of minutes and seconds. $0.4567 \times 60 = 27.402$ so there are 27 arcmin and some number of seconds. $0.402 \times 60 = 24.12$ so there are 24.12 arcsec (24 to the nearest whole number). Therefore the angle is 3° 27′ 24″.

In some circumstances the degree/arcmin/arcsec system of units for measuring angles is not the most convenient, and angles are expressed in **radians** instead. One radian is defined as shown in Figure 6.5: it is the angle between two lines drawn from the centre of a circle such that the length of *arc* between the ends is equal to the radius.

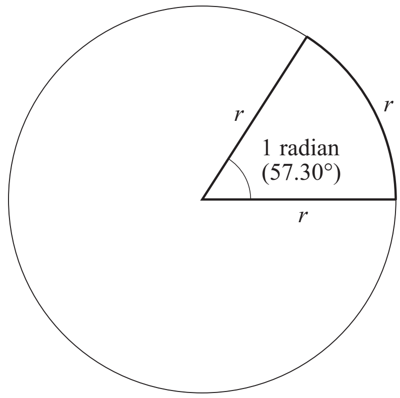

Figure 6.5 Defining the radian.

From this definition, we generalize to any angle:

$$\text{angle measured in radians} = \text{arc length/radius} \tag{6.40a}$$

In symbols

$$\theta/\text{rad} = l/r \tag{6.40b}$$

where θ is the Greek letter theta, often used to represent angles, and l is the length of the arc. Notice that the equation includes the SI unit of radians, often abbreviated to rad.

A complete turn covers an arc length of $2\pi r$ (the *circumference* of the circle) so this corresponds to 2π radians. We can therefore write

$$360° = 2\pi\ \text{radians} \tag{6.41}$$

so $$1\ \text{radian} = 360°/2\pi = 57.30° \tag{6.42}$$

and $$1° = 2\pi\ \text{radians}/360 = 0.017\,45\ \text{radians} \tag{6.43}$$

Angles measured in radians are sometimes expressed using fractions or multiples of the number π, but not necessarily so. These examples show how to convert between radians and degrees.

EXAMPLE 6.12

What is a right angle expressed in radians?

A right angle is one-quarter of a complete revolution, so it can be written as 2π rad/4 = $\pi/2$ radians = 1.571 radians. You could get the same answer using Equation 6.43, namely $90 \times 0.017\,45$ radians = 1.571 radians.

EXAMPLE 6.13

A certain angle is written as $\pi/6$ radians. What is this in degrees?

From Equation 6.41, $\pi/6$ radians must correspond to one-twelfth of a complete turn, that is 30°. Alternatively, from Equation 6.42, we have $(360°/2\pi) \times \pi/6 = 30°$.

EXAMPLE 6.14

What is an angle of 1 arcmin expressed in radians?

1 arcmin = 1°/60, and from Equation 6.43, we have (2π radians/360) × (1/60) = 2.909×10^{-4} radians.

QUESTION 6.27

What is 60° expressed in radians? Express your answer as a multiple or fraction of π and as a decimal number of radians.

QUESTION 6.28

What is π/4 radians expressed in degrees?

QUESTION 6.29

What is an angle of 1 arcsec expressed in radians?

6.5.3 Right-angled triangles

The branch of maths called **trigonometry** deals with relationships between angles and distances so it is very useful in astronomy and planetary science. Many of the important relationships in trigonometry can be illustrated via right-angled triangles, that is triangles that contain one angle of 90°. Figure 6.6 shows a right-angled triangle. Notice the 'square' symbol used to label the right angle. One of the other angles is labelled with the *Greek letter* theta (θ). Since the angles inside *any* triangle always add up to 180°, the unlabelled angle must be equal to $90° - \theta$.

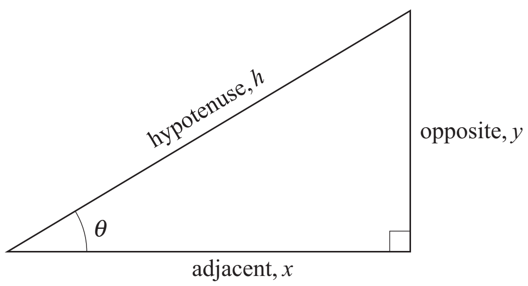

Figure 6.6 A right-angled triangle.

The longest side of a right-angled triangle, opposite the right angle, is called the hypotenuse. The two shorter sides that are opposite and adjacent to (next to) the angle we are interested in (θ) are called the opposite and adjacent sides.

The sides of a right-angled triangle have a useful property, known as **Pythagoras' theorem**, after the Greek mathematician and philosopher who lived in the 6th century BC. Representing the length of the hypotenuse by h and the other two sides by x and y as shown in Figure 6.6,

$$x^2 + y^2 = h^2 \tag{6.44}$$

If we know the lengths of two sides we can always work out the third. For example, if $x = 3$ m and $y = 4$ m, then $x^2 + y^2 = 9\,\mathrm{m}^2 + 16\,\mathrm{m}^2 = 25\,\mathrm{m}^2$. So $h^2 = 25\,\mathrm{m}^2$ and $h = 5$ m. The numbers don't always work out so neatly, but Equation 6.44 applies to *all* right-angled triangles. Notice that you need to square each length separately before adding them together, and then find the *square root*.

EXAMPLE 6.15

A certain right-angled triangle has sides $x = 8.5$ cm, $y = 5.0$ cm. How long is its hypotenuse?

From Equation 6.44,

$$h^2 = (8.5\,\text{cm})^2 + (5.0\,\text{cm})^2 = 72.25\,\text{cm}^2 + 25.0\,\text{cm}^2 = 97.25\,\text{cm}^2$$

so $h = \sqrt{97.25\,\text{cm}^2} = 9.9\,\text{cm}$

QUESTION 6.30

The hypotenuse of a certain right-angled triangle is 7.2 cm long. One of the other sides measures 6.5 cm. How long is the third side?

6.5.4 Trigonometric functions

A second useful property of triangles concerns their angles as well as their sides. The relative lengths of the sides of a triangle depend *only* on the angles enclosed. Put another way round, the angles in the triangle depend *only* on the relative lengths of the sides. In a right-angled triangle, the ratios between the sides are given the names: **sine**, **cosine** and **tangent** often abbreviated to sin, cos and tan. For any angle θ in a right-angled triangle (see Figure 6.6)

$$\sin\theta = \text{opposite/hypotenuse} = y/h \qquad (6.45)$$

$$\cos\theta = \text{adjacent/hypotenuse} = x/h \qquad (6.46)$$

$$\tan\theta = \text{opposite/adjacent} = y/x \qquad (6.47)$$

These three trigonometric functions are stored in a calculator. For example, to find the sine of 30°, you need to key in sin 30, or 30 sin (depending on how your calculator operates) and the calculator will display 0.5. Notice that the sin and cosine of any angle are never greater than 1 because the two sides x and y are always less than h, but the tangent of an angle can have *any* value.

EXAMPLE 6.16

A surveyor stands 20 m from a tower and finds that the angle from the top of the tower to the ground is 60°. See Figure 6.7. How tall is the tower?

Compare Figures 6.6 and 6.7. Put distance to tower = x = 20 m, and height = y, then use Equation 6.47:

$$y = x\tan\theta = 20\,\text{m} \times \tan 60° = 35\,\text{m}$$

(First use a calculator to find tan 60°, then multiply by 20.)

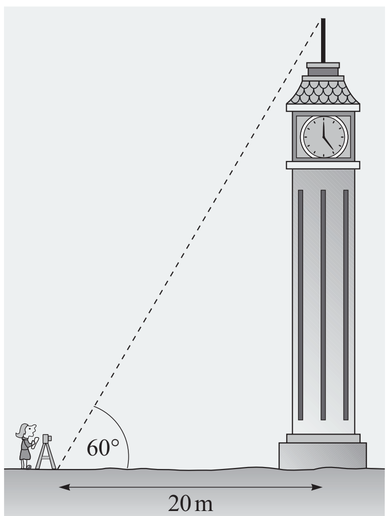

Figure 6.7 Using trigonometry to find the height of a tower.

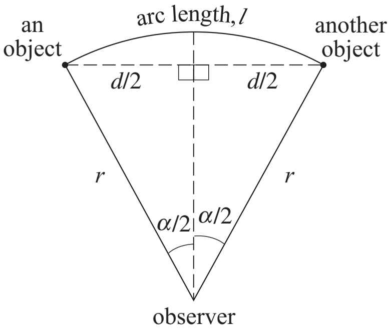

Figure 6.8 Two objects separated by an angle α.

EXAMPLE 6.17

An astronomer observes two objects at a distance r. The angle between the directions to the objects is α. Use Figure 6.8 to derive an equation relating the objects' separation, d, to their distance and the angle α.

Work with one of the right-angled triangles containing the angle $\alpha/2$. The length of the hypotenuse is r and the side we want to know, $d/2$, is opposite the angle $\alpha/2$, so we can use Equation 6.45. Put $\theta = \alpha/2$, $y = d/2$ and $h = r$

$$\sin(\alpha/2) = d/2r$$

multiply by $2r$:

$$d = 2r\sin(\alpha/2) \qquad (6.48)$$

Notice the brackets around $\alpha/2$ to ensure that we find the sine of half the angle α (which is *not* the same as finding $\sin\alpha$ then halving the result, so the 2s do *not* cancel out).

QUESTION 6.31

Use one of Equations 6.45, 6.46 or 6.47 to find the distance from the top of the tower to the place where the surveyor is standing in Figure 6.7.

Take care, when obtaining the sine, cosine or tangent of an angle, that the units of the angle match those of the calculating procedure. Most calculators can be switched to receive angles in *radians* or rather than degrees (try pressing a button marked DRG, or consult the instructions for your calculator).

EXAMPLE 6.18

The angle in Figure 6.7 could also be labelled π/3 radians. To work out the height of the tower, use the same method as Example 6.17. Use your calculator to work out π/3 (= 1.047…) then make sure it is in 'radian mode' and press tan to get tan(π/3 rad) = 1.732…. Then multiply by 20 to get the answer 35 m as in Example 6.17.

QUESTION 6.32

Use $\theta = \pi/3$ radians in one of Equations 6.45, 6.46 or 6.47 to find the distance from the top of the tower to the place where the surveyor is standing in Figure 6.7.

6.5.5 Small angles

Astronomers often need to deal with angles that are very small, for example when they observe objects whose size or separation is small compared with their distance. In Figure 6.9, if the angle θ is small, the distance d becomes very similar to the arc length l. So if θ is measured in *radians*, we can write

$$d/r \approx l/r = \theta/\text{rad} \qquad (6.49)$$

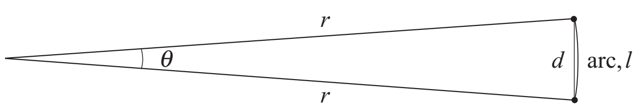

Figure 6.9 Small angle approximation.

This **small angle approximation** enables us to find d very easily using r and the angle θ just by putting θ in radians with no need for trigonometric functions.

The small angle approximation leads to some further useful approximations involving trigonometric functions. From triangle OAC in Figure 6.10, when θ is small

$$\sin\theta = \mathrm{AC}/r \approx l/r = \theta/\mathrm{rad} \quad (6.50)$$

and $\cos\theta = \mathrm{OA}/r \approx r/r = 1$ (6.51a)

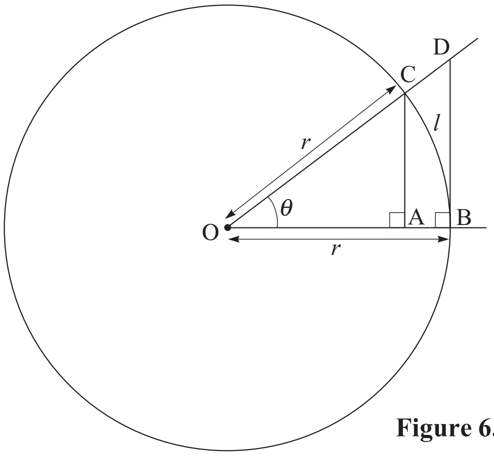

Figure 6.10 Trigonometric functions of the angle θ.

From triangle OBD in Figure 6.10 when θ is small

$$\tan\theta = \mathrm{BD}/r \approx l/r = \theta/\mathrm{rad} \quad (6.52)$$

and $\cos\theta = r/\mathrm{OD} \approx r/r = 1$ (6.51b)

So from Equations 6.50 and 6.52 we have, for small θ

$$\sin\theta \approx \tan\theta \approx \theta/\mathrm{rad} \quad (6.53)$$

The range of angles for which we can use the small angle approximation depends on how precise we want to be. For an angle of 1° (1.745×10^{-2} rad) the results using the approximation are correct to at least three significant figures, and the approximation gets even better for smaller angles.

EXAMPLE 6.19

Viewed from Earth, the Sun has an angular diameter α close to 0.5° (the angle between two lines pointing towards opposite edges of the Sun). Given that its distance from Earth is $r = 1.50 \times 10^{11}$ m, what, approximately, is its actual diameter?

From Equation 6.42, $0.5° = 8.7 \times 10^{-3}$ rad. Then from Equation 6.49, diameter $d = \alpha r = 8.7 \times 10^{-3} \times 1.50 \times 10^{11}\ \mathrm{m} = 1.3 \times 10^{9}\ \mathrm{m}$.

EXAMPLE 6.20

Use results from the small angle approximation to simplify Equation 6.48, which was derived in Example 6.18 for the separation of the two objects in Figure 6.8.

Using Equation 6.53

$$d = 2r\sin(\alpha/2) \approx 2r(\alpha/2)/\mathrm{rad} = r\alpha/\mathrm{rad}$$

From Figure 6.8, $\alpha/\mathrm{rad} = l/r$ so $d = \alpha l$, as expected from Equation 6.49.

QUESTION 6.33

The Moon's diameter is $d = 3.5 \times 10^6$ m. Its angular diameter, viewed from Earth, is approximately 0.5°. What is its distance from Earth?

6.6 Graphs

6.6.1 Plotting and reading graphs

Table 6.1 Data for Figure 6.11.

Time t/s	Speed v/m s^{-1}
0.0	3.0
1.0	5.0
3.0	9.0
4.0	11.0

A **graph** illustrates how two quantities are related to one another by displaying points corresponding to pairs of values. The points are often joined by a straight line or a smooth curve. For example, Figure 6.11 is a graph of an object's speed, v, plotted against time, t. In this example, the object was already moving at *speed* $u = 3.0\,\mathrm{m\,s^{-1}}$ when $t = 0.0$ s, and its *acceleration* was $a = 2.0\,\mathrm{m\,s^{-2}}$. Table 6.1 shows the values of v calculated at various times using the equation

$$v = u + at \tag{6.54}$$

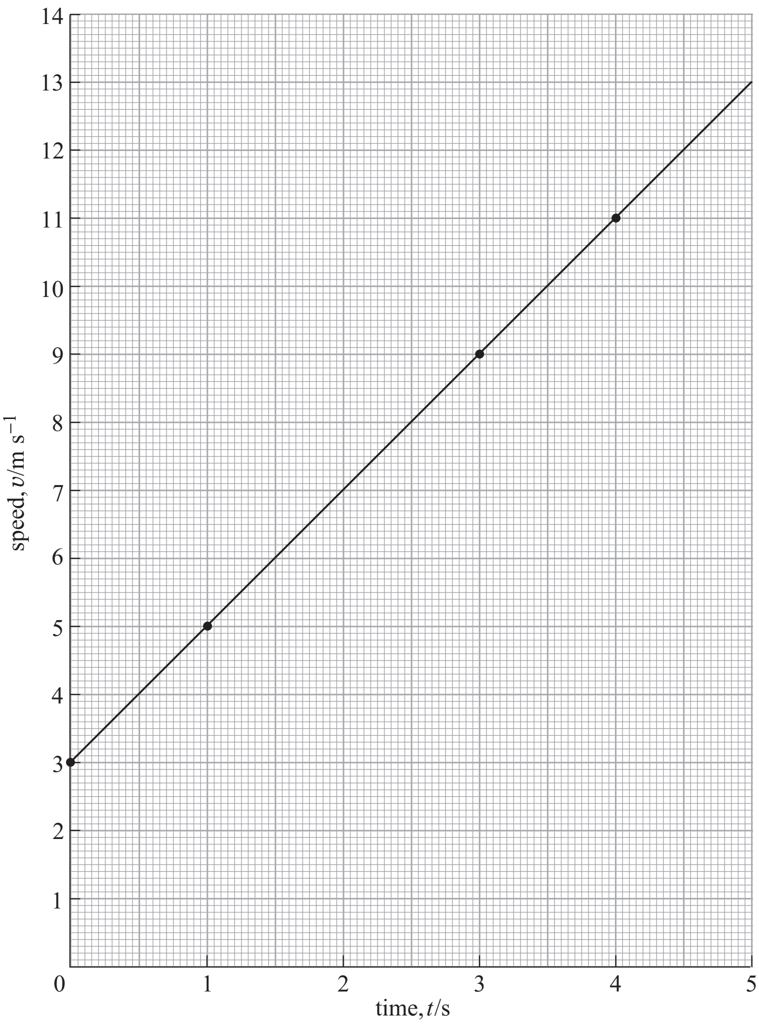

Figure 6.11 A graph of speed against time.

Notice that the graph's **axes** (the lines that show where the numbers on the graph should be plotted) have a **linear scale** — that is, they are marked at equal intervals even though the values in Table 6.1 are not evenly spaced, and are labelled in the form 'quantity/unit'. The axis running across the page is generally called the *x*-axis, and the one running vertically is the *y*-axis, even though the quantities plotted on them quite often have symbols other than *x* and *y*. We often say that values of the quantity on the *y*-axis are 'plotted against' corresponding values of the quantity on the *x*-axis. In Figure 6.11, the four plotted points are marked with dots and, because in this case they all lie on a straight line, are joined with a straight line.

The **origin** of a graph is the point corresponding to zero on both axes, and this is usually drawn at the bottom left of the graph where the axes cross. Sometimes, if we want to plot a small range of values that lie very far from zero, we choose to draw a graph whose axes do *not* start at the origin. Provided the axes are clearly labelled, this is fine. Figure 6.12 gives an example where it would clearly be silly to start with the year zero on the *x*-axis, and if we started at zero on the *y*-axes then all the values would be squashed to the top of the graph making them very hard to read.

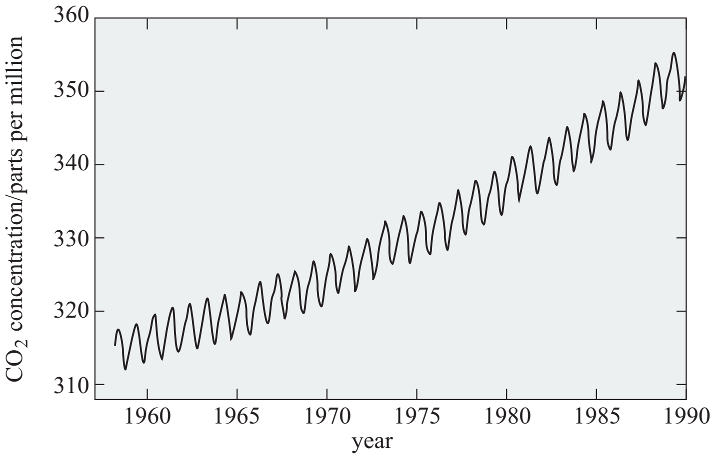

Figure 6.12 A graph showing variations in atmospheric carbon dioxide. An example of a graph that does not include the origin.

In plotting a graph, the convention is usually that the **independent variable** is plotted along the *x*-axis and the **dependent variable** up the *y*-axis. The independent variable is the quantity that, in an experiment or measurement, we start off by knowing, and the dependent variable is the quantity that we set out to measure or calculate. For example, if we ask 'how does the speed of this object change with time?' then time is the independent variable and speed the dependent variable. Similarly, if we ask 'how does the lifetime of a star depend on its mass?' then mass is the independent variable and we would probably plot a graph of mass along the *x*-axis and lifetime up the *y*-axis. But this is a convention not a hard-and-fast rule, and sometimes it is desirable to plot the axes the other way round. For example, Figure 6.13 shows how the temperature of the Earth's atmosphere depends on altitude. Altitude is the independent variable, but here it is plotted vertically because that is the direction in which altitude is measured.

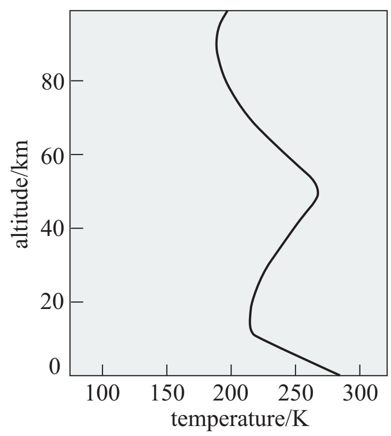

Figure 6.13 The variation of atmospheric temperature with height above the Earth's surface.

A graph can convey information in a much more compact way than a table of data. From Figure 6.11 you can deduce the object's speed at times that are not listed in Table 6.1. For example, by **interpolating** between the plotted points (reading

between them) you can see that at a time of 1.5 s the speed was 6 m s^{-1}, and similarly you can deduce that the object reached a speed of 8.5 m s^{-1} at a time t = 2.7 s. And by **extrapolating** (extending) the graph beyond the plotted points you can work out that at 4.5 s the speed would be 12 m s^{-1} (provided the object continued to move with the same acceleration).

The scales of the axes in Figure 6.11 were chosen to make plotting and reading the graph as easy as possible. This means using as much of the available space as possible while at the same time making sure that the graph-paper squares correspond to 'easy' numbers on the axes. A scale of one, two or five large graph-paper squares to each division on the axes will enable you to plot and read the graph easily. *Avoid* the temptation to have four, three or some other number of graph-paper squares per division, as the subdivisions on your scale will then fall in awkward places and you will need to do some arithmetic every time you plot or read a point.

6.6.2 Gradient of a graph

A graph's **gradient** is defined as shown in Figure 6.14 as

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (6.55\text{a})$$

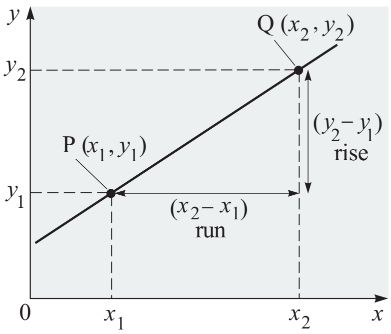

Figure 6.14 The gradient of a line joining two points P and Q.

A useful way to think of this is

$$\text{gradient} = \frac{\text{rise}}{\text{run}} \qquad (6.55\text{b})$$

In Figure 6.14, notice that the two points P and Q can lie anywhere on the line. To increase the precision of a calculation using values of x and y read from the graph, the two points should be as widely separated as possible. To work out the gradient, use values of x and y read from the scales of the graph (which will *not* in general be the same as distances measured off the graph with a ruler).

EXAMPLE 6.21

The graph in Figure 6.15 shows the distances to various earthquake monitoring stations plotted against the time at which waves arrived following a particular earthquake. Calculate the gradient of the graph and suggest a physical interpretation for its value.

Using the two points at time = 2 s, distance = 0 km and, time = 30 s, distance = 160 km, from Equation 6.55 we have

$$\text{gradient} = \frac{160\,\text{km}}{(30-2)\,\text{s}} = 5.7\,\text{km}\,\text{s}^{-1}$$

The gradient was found by dividing a distance by a time interval. In this graph, the gradient is equal to the speed of the waves.

The calculation of a gradient includes units. Often the gradient of a graph has a physical meaning, and in Example 6.22 the gradient of the graph was equal to the wave speed (5.7 km s^{-1}). The gradient of *any* graph of distance versus time will be interpreted as a speed, and in a graph of speed versus time the gradient is equal to the acceleration.

Notice that in Figure 6.15 the dependent variable is distance, but the graph is plotted as distance against time because that gives a more direct physical interpretation for the gradient.

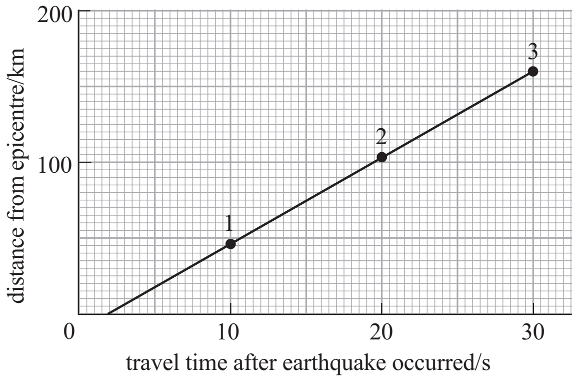

Figure 6.15 Graph of distance against time for waves produced in an earthquake.

QUESTION 6.34

What is the gradient of the graph shown in Figure 6.11? Suggest how the gradient of this graph can be interpreted physically.

Not all graphs are straight lines. For example, Figure 6.16 shows how the radius R of a hypothetical star might increase with time. It is still possible to calculate the gradient of a curved graph, as Figure 6.16 shows. Draw a straight line that just touches the curve at the point you are interested in, then calculate the gradient of that line. (In practice it is often difficult to judge exactly where to draw the line so the gradient cannot be calculated very precisely.)

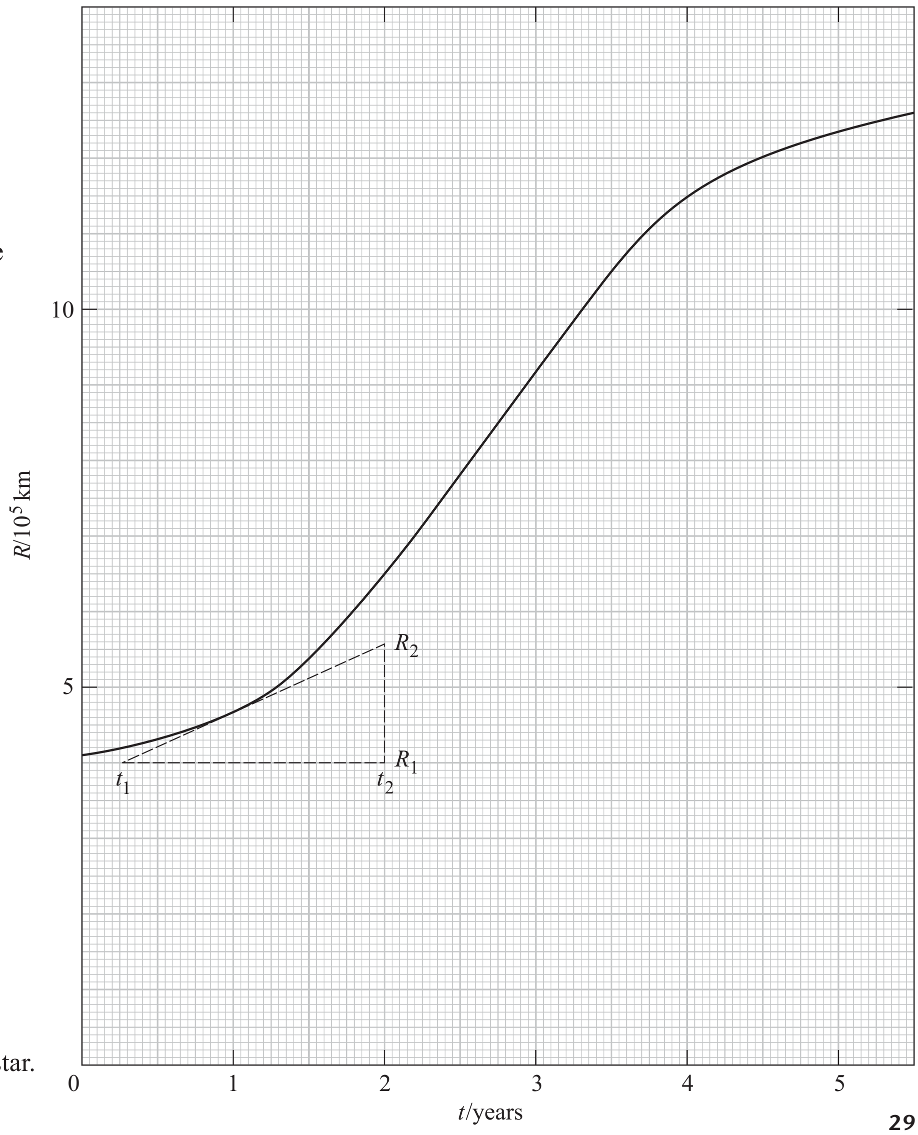

Figure 6.16 Expansion of a hypothetical star.

EXAMPLE 6.22

Use the gradient of the graph in Figure 6.16 to find the outward speed of the star's surface at a time of 1 year.

From Figure 6.16 and Equation 6.55, rise = $R_2 - R_1 = (5.6 - 4.0) \times 10^5$ km, and run $= t_2 - t_1 = (2.00 - 0.25)$ yr $= 1.6 \times 10^5$ km/1.75 yr $= 0.9 \times 10^5$ km yr^{-1}.

QUESTION 6.35

Calculate the gradient at $t = 4$ yr in Figure 6.16.

6.6.3 Equation of a straight line

If any quantity y is *directly proportional* to some other quantity x,

$$y \propto x, \text{ or } y = kx \tag{6.56}$$

then a graph of y against x is a straight line through the origin as shown in Figure 6.17. The *gradient* of such a line is always equal to the *constant of proportionality*, k. Starting from the *origin*, the 'rise' to any point is equal to its value of y, and the corresponding 'run' is equal to its value of x, and so from Equations 6.55 and 6.56,

$$\text{gradient} = y/x = k$$

In the example shown in Figure 6.17a, the gradient is 1.5 (e.g. rise/run = 6/4 = 1.5). Any value of y can be calculated from x using the equation

$$y = 1.5x$$

In Figure 6.17b, the line from Figure 6.17a has been shifted upwards by a distance labelled c (here $c = 2$). Now to calculate any value of y we need to use the

$$y = 1.5x + 2$$

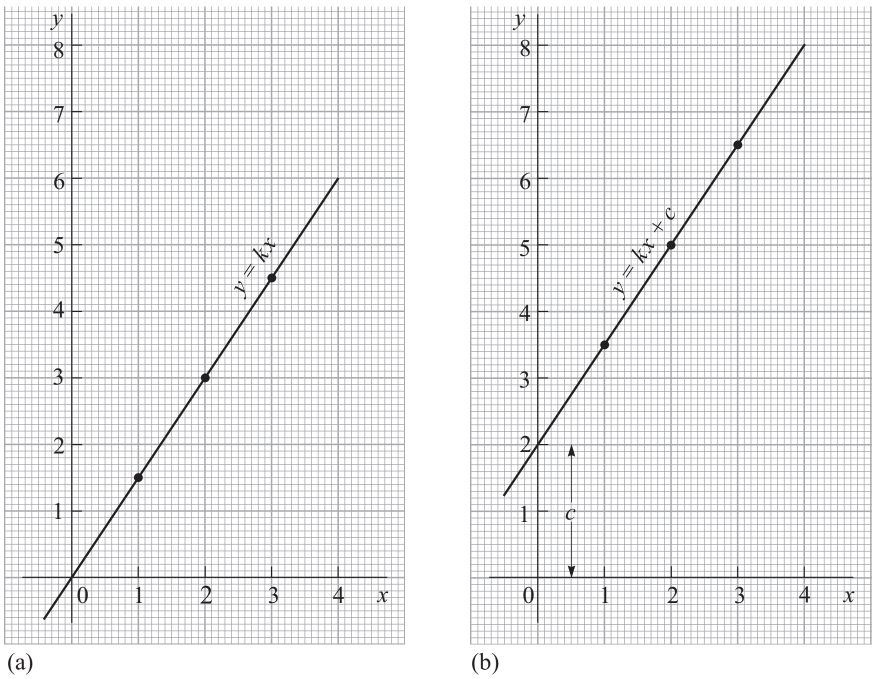

Figure 6.17 A graph of y against x when (a) $y = kx$ (b) $y = kx + c$.

The graph in Figure 6.17b still has a gradient of 1.5 since rise/run = (8 – 2)/4 = 6/4 = 1.5, but the line now cuts the *y*-axis above the origin at *y* = 2. Since this line does *not* pass through the origin, *y* is *not* proportional to *x*.

The **general equation of a straight line** is

$$y = kx + c \qquad (6.57)$$

This tells us how to calculate the value of *y* corresponding to any value of *x*. And when those values are plotted on a graph, the gradient is equal to *k*, and the line cuts the *y*-axis above the origin when $y = c$; this value of *y*, corresponding to $x = 0$, is often known as the **intercept** on the *y*-axis. In many situations the intercept and gradient both have some physical meaning. For example, if an object's speed *v* at some time *t* is given by the equation

$$v = u + at = at + u$$

Comparison with Equation 6.57 shoes that a graph of *v* plotted against *t* has gradient equal to the *acceleration*, *a*, and intercepts the *y*-axis at *u*, which is the speed when $t = 0$.

6.6.4 Logarithmic scales

Sometimes we need to plot graphs covering a huge range of values. For example, the *temperature* in the outer regions of the Sun varies with height above the surface as shown (approximately) in Table 6.2. If we want to show details of variation close to the surface we might plot a graph with axes as shown in Figure 6.18a, but then points representing the larger values would need a sheet of paper at least one hundred times wider and one thousand times longer. If we wanted to include all the values on a graph of reasonable size, then we might plot a graph with axes as shown in Figure 6.18b. Now we can plot the larger values, but the smaller ones are crammed so close to the origin that they cannot be seen. In either case, a *linear scale* is unsatisfactory.

Table 6.2 Variation of temperature with height in the outer regions of the Sun.

Height above surface, h/km	Temperature, T/K
1.0×10^2	6.0×10^3
3.0×10^2	5.0×10^3
1.0×10^3	6.0×10^3
3.0×10^3	2.0×10^5
1.0×10^4	1.0×10^6
1.0×10^5	2.0×10^6

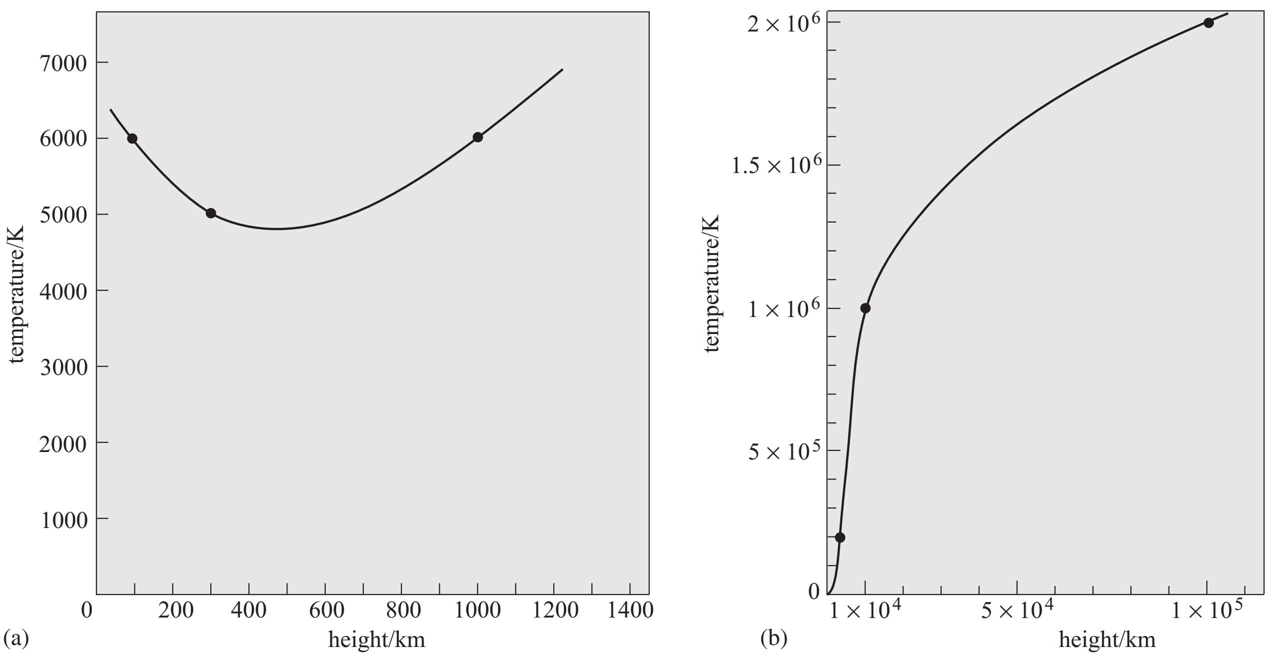

Figure 6.18 Two possible choices of axes for plotting the values from Table 6.2.

Table 6.3 Logarithms of the values listed in Table 6.2.

$\log_{10}(h/\text{km})$	$\log_{10}(T/\text{K})$
2.0	3.8
2.5	3.7
3.0	3.8
3.5	5.3
4.0	6.0
5.0	6.3

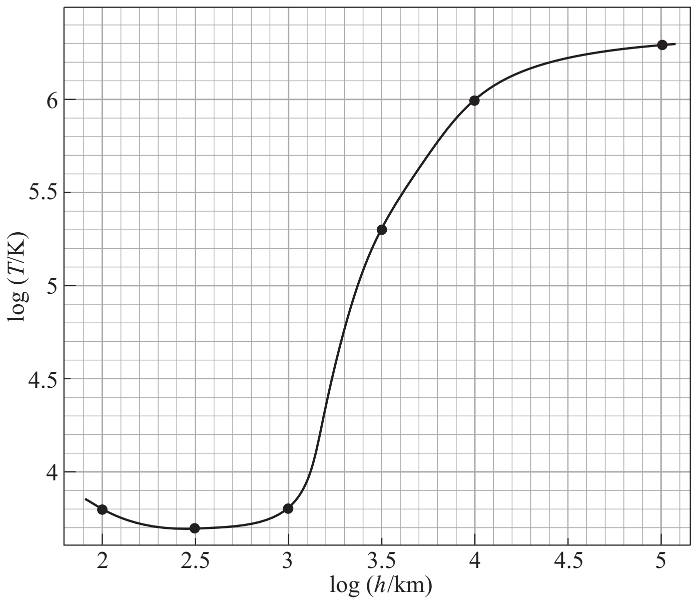

Figure 6.19 A logarithmic graph showing the variation of temperature with height in the outer regions of the Sun.

To display values where the largest is more than few tens of times bigger than the smallest, we often plot a graph with a **logarithmic scale** (often abbreviated to 'log scale'). There are two ways to construct a logarithmic scale, which both boil down to exactly the same thing.

One way to make a logarithmic scale is to label the axes of a linear scale using the *common logarithms* of the quantities to be plotted. Table 6.3 lists the logarithms of the values given in Table 6.2. The logarithms cover a much smaller range than the values themselves. Figure 6.19 shows a logarithmic graph of the data from Table 6.3.

The other way to make a log scale is to label the axes using *powers of ten*, in such a way that the powers increase in equal steps. When plotting a graph in this way, we can use logarithmic graph paper as shown in Figure 6.20. The beginning of each 'cycle' is

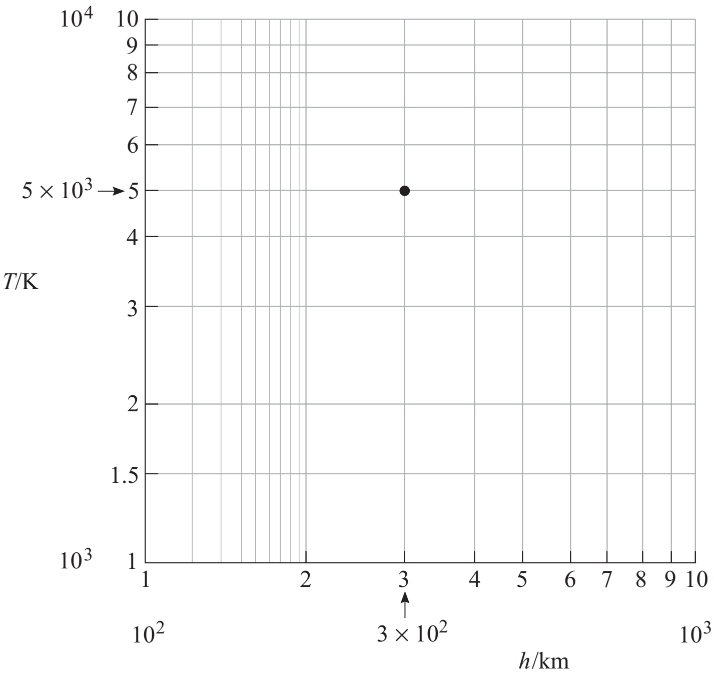

Figure 6.20 Plotting a point on logarithmic graph paper.

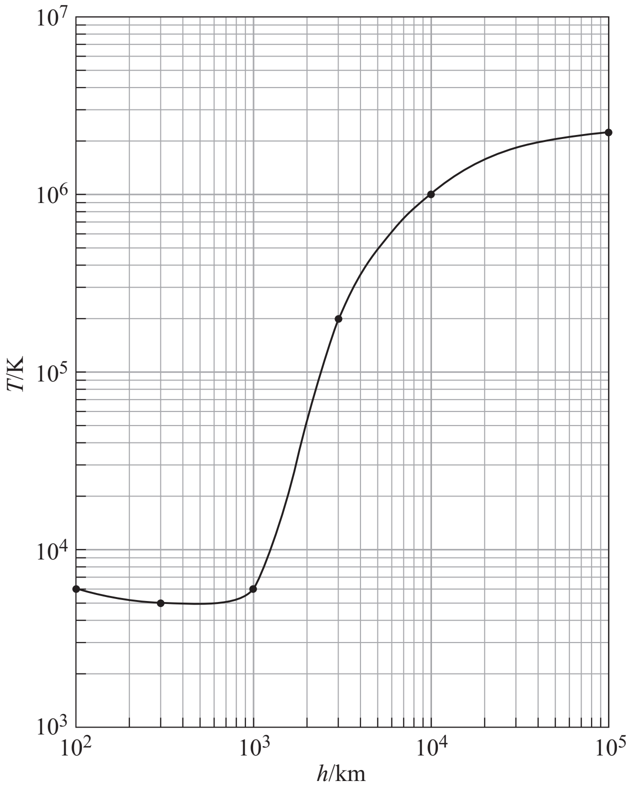

Figure 6.21 A graph of data from Table 6.2 plotted on logarithmic graph paper.

labelled with a whole-number power of ten, and the grid shows where to plot intermediate values. Figure 6.20 shows how to plot the point for $h = 3.0 \times 10^2$ km, $T = 5.0 \times 10^3$ K on logarithmic graph paper, and Figure 6.21 shows a graph of the data from Table 6.2 plotted in this way.

Notice that Figure 6.21 shows a graph with exactly the same shape as that in Figure 6.19. Either way of plotting the graph is equally good, because both display the information in the same way.

Logarithmic graphs can involve negative *powers of ten* (and hence negative *logarithms*) as well as positive ones. There is an example in Figure 6.22, which shows the *frequencies* and *wavelengths* of *electromagnetic radiation*. The numbers on such a scale get smaller and smaller with each power of ten, but never quite reach zero, so the axes of a logarithmic graph do not include the *origin*.

The examples in Figures 6.19 and 6.21 are **log–log graphs**; both axes have logarithmic scales. Sometimes it can be useful to plot a log–linear graph, in which only one axis has a logarithmic scale and the other has an ordinary *linear scale*. Such a **log–linear graph** might be appropriate if only one set of values covers a large range.

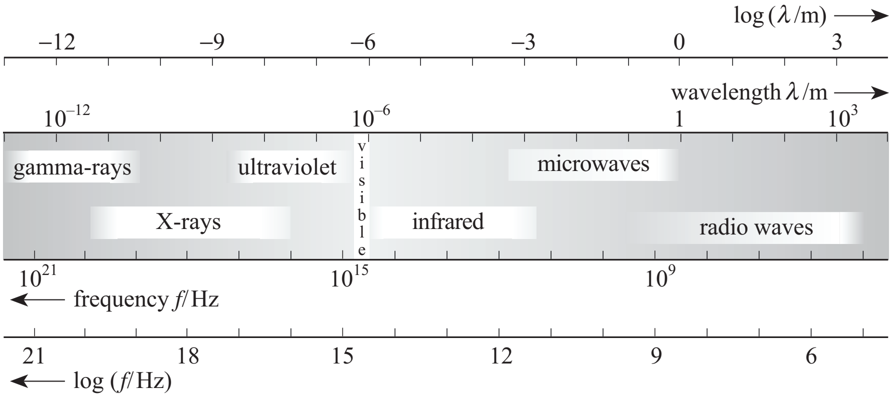

Figure 6.22 The electromagnetic spectrum.

QUESTION 6.36

Table 6.4 lists some information about the planets in the Solar System: their approximate distances from the Sun and their orbital periods (the time to travel once around an orbit). Plot two logarithmic graphs of these values, using the grids in Figure 6.23. (*Hint*: start by making a table of $\log_{10}(d/10^6\,\text{km})$ and $\log_{10}(T/\text{yr})$.)

Table 6.4 The distances and orbital periods of planets in the Solar System.

Planet	Distance from Sun $d/10^6$ km	Orbital period T/yr
Mercury	58	0.24
Venus	1.1×10^2	0.62
Earth	1.5×10^2	1.0
Mars	2.3×10^2	1.9
Jupiter	7.8×10^2	12
Saturn	1.4×10^3	29
Uranus	2.9×10^3	84
Neptune	4.5×10^3	1.6×10^2
Pluto	5.9×10^3	2.5×10^2

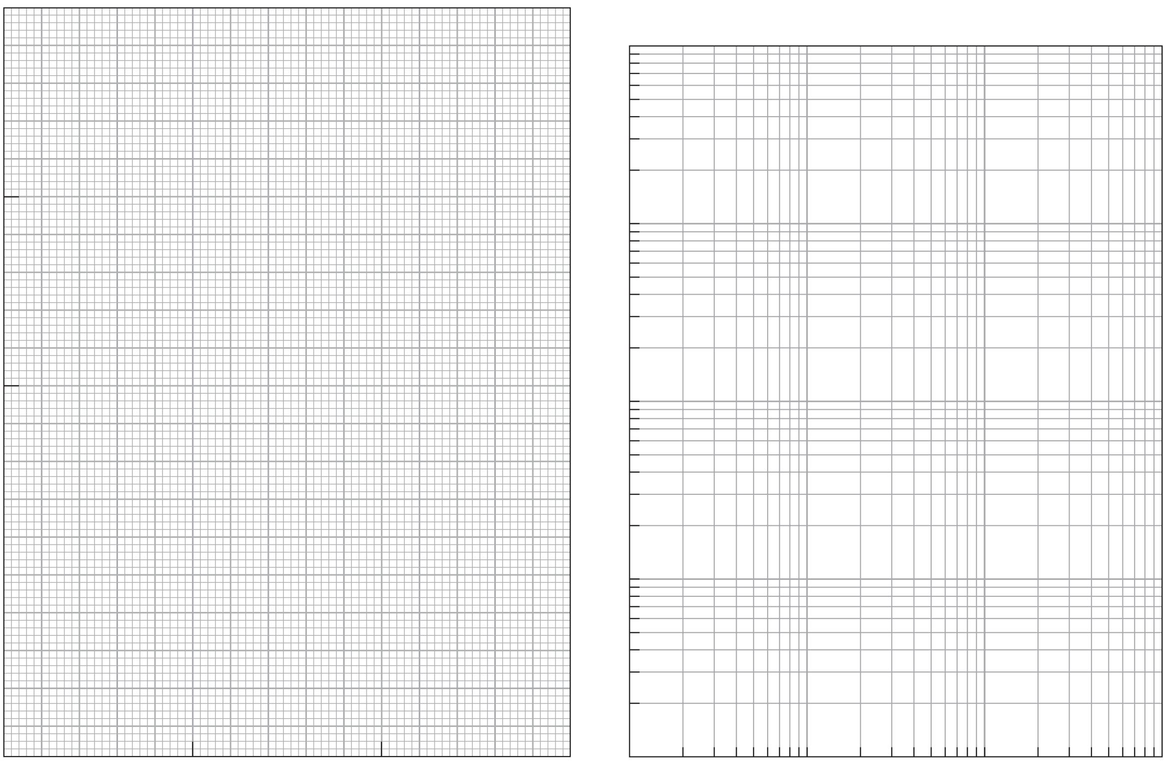

Figure 6.23 Graph paper for use with Question 6.36.

6.6.5 Reading log scales

To read information from a graph plotted on logarithmic graph paper, simply use the grid lines and the 'powers of ten' labels. If the graph is plotted using logarithms of values plotted on a linear scale, then read the graph in the normal way then find the *antilog* of the values. For example, if you wanted to use Figure 6.19 to find the height at which the temperature was 1×10^5 K, you would first find

$$\log_{10}(T/\text{K}) = \log(1 \times 10^5) = 5.0$$

then read the graph as shown in Figure 6.24. Here $\log_{10}(h/\text{km}) \approx 3.44$

so $h/\text{km} = \text{antilog}_{10}(3.44) = 2.75 \times 10^3$

i.e. $h = 2.75 \times 10^3$ km

You will probably come across logarithmic graphs plotted using powers of ten, but where divisions are not shown. Figure 6.24 shows such a graph based on Figure 6.21. This makes it look quite tricky to read values lying between the powers of ten.

One way to read a graph such as Figure 6.24 is to imagine the axes labelled with logarithms as in Figure 6.19. Then estimate roughly where your chosen point lies in relation to whole-number powers. For example, if you needed to use Figure 6.24 to find the height at which the temperature was 1×10^5 K, you might judge that the relevant point lay about half of the way along from 10^3 km to 10^4 km. So you would then say that

$$\log_{10}(h/\text{km}) \approx 3.5$$

and, using antilogs as above, could work out that

$$h/\text{km} \approx \text{antilog}(3.5) \approx 3 \times 10^3$$

$$h \approx 3 \times 10^5 \text{ km}$$

This is of course a less precise answer than you would get if the grid lines were shown.

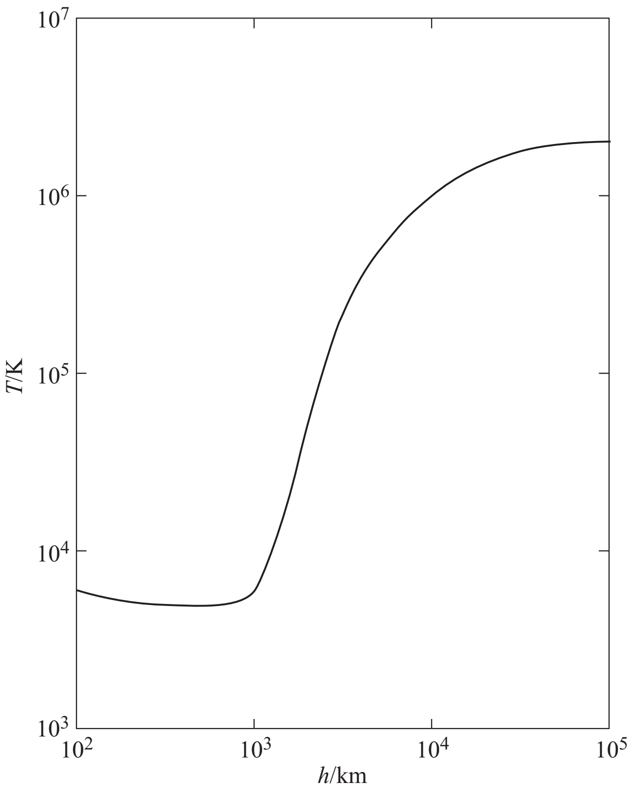

Figure 6.24 A logarithmic graph drawn without grid lines, or sub-divisions on the *x*-axis.

Another way to read graphs such as that in Figure 6.24 is to picture the grid lines from Figure 6.21. Notice that the grid lines on logarithmic graph paper get closer together as you move towards the higher power of ten, and the '3' falls about half way along. This gives us a quick approximate way to read the scale. A point lying mid-way between two powers of ten can be read as '$3 \times \ldots$' the lower power, and other points can likewise be estimated. '$1.5 \times \ldots$' lies about one-fifth of the way along, '$2 \times \ldots$' lies about one-third of the way along, and '$5 \times \ldots$' lies about three-quarters of the way towards the higher power.

QUESTION 6.37

On Figure 6.19, find the point on the x-axis corresponding to a height of 5×10^3 km. Then find the temperature at that height.

QUESTION 6.38

On Figure 6.24, mark the point corresponding to about 3×10^4 km. Then read the approximate temperature at that height.

6.7 Answers and comments for Topic 6

QUESTION 6.1

(i) $15a^7$ (ii) $3b^3$ (iii) 6 (iv) $16d^8$ (i.e. $2^4 \times d^8$)

QUESTION 6.2

(i) $2a^{-4}$ (ii) $b^{-2}/3$

QUESTION 6.3

(i) 0.008 (ii) 0.028 57…

QUESTION 6.4

(i) $6a^{3/4}$ (i.e. $2a^{3/2} \times 3a^{1/2}$) (ii) $3\sqrt{b}$ (iii) $2c/d$

QUESTION 6.5

(i) 3.162… (ii) 4913 (iii) 4.429…

QUESTION 6.6

(i) 1.5×10^{11} m (ii) 1.60×10^{-19} C

QUESTION 6.7

(i) 3.03×10^{-19} (ii) 6.17×10^{14}

QUESTION 6.8

(i) 4 (ii) −5 (iii) 0

QUESTION 6.9

(i) 0.3010… (ii) 0.501 371… (iii) −0.204…

QUESTION 6.10

The answers to this question are exactly the same as those for Question 6.9.

(i) If $10^x = 2$ then $x = \log_{10}(2) = 0.3010…$ (ii) 0.501 371… (iii) −0.204…

QUESTION 6.11

(i) 3.162… (ii) 16.982… (iii) 0.0588…

QUESTION 6.12

(i) two (ii) two (iii) two (iv) three (v) four

QUESTION 6.13

On a calculator,

Speed = $241 \times 10^6\,\text{m}/(33 \times 3600\,\text{s}) = 2.028\,62 \times 10^3\,\text{m}\,\text{s}^{-1}$

The time, 33 hours, has only two significant figures so the final answer only has two: the average speed is $2.0 \times 10^3\,\text{m}\,\text{s}^{-1}$.

QUESTION 6.14

(i) $10^{-23}\,\text{J}\,\text{K}^{-1}$ (ii) 10 (i.e. 10^1) (iii) $10^{30}\,\text{kg}$ (iv) $10^{12}\,\text{m}$

QUESTION 6.15

Distance $\approx (3 \times 3 \times 10^8 \times 10^7)\,\text{m} = 9 \times 10^{15}\,\text{m}$.

QUESTION 6.16

Write $ma = F$ then divide both sides by m to get $a = F/m$.

QUESTION 6.17

First use one of the two methods of Example 6.8 to get r^2 on its own.

Either start by taking the reciprocal of both sides to get $r^2/GMm = 1/F$, then multiply by G, M and m to get $r^2 = GMm/F$,

or multiply both sides by r^2 to get $r^2F = GMm$ then divide by F to get $r^2 = GMm/F$.

Finally take the square root of both sides to get

$$r = \sqrt{\frac{GMm}{F}}$$

QUESTION 6.18

Distance $x = 2\pi r$ so $v = 2\pi r/T$.

QUESTION 6.19

The mass m is the unwanted quantity. First make m the subject of the density equation: $m = \rho V$. Then substitute for m in the equation for internal energy: $\Delta q = \rho Vc\,\Delta T$.

QUESTION 6.20

Pressure $p \propto n$ and $p \propto T$. (Notice that $p \propto k$ is *not* a correct answer. It is a meaningless statement because k is a constant.)

QUESTION 6.21

(i) $y \propto 1/x$ (ii) $y \propto x^2$ (iii) $y \propto 1/x^2$

QUESTION 6.22

We can first write $L \propto kAT^4$ or $L = kAT^4$ where k is some constant. Then, because $A = Kr^2$ where A is some constant, we can also write $L = Kkr^2T^4$ which becomes $L \propto r^2T^4$.

QUESTION 6.23

First write $t^2 = kr^3$. Then use t_1 and r_1 and t_2 and r_2 to represent quantities for the Earth and for Jupiter respectively.

$$t_1^2 = kr_1^3 \text{ and } t_2^2 = kr_2^3$$

Either make k the subject to get

$$\frac{t_2^2}{r_2^3} = \frac{t_1^2}{r_1^3}$$

or divide one version by the other to get

$$\left(\frac{t_2}{t_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3$$

In either case,

$$\begin{aligned} t_2 &= t_1\sqrt{\left(\frac{r_2}{r_1}\right)^3} \\ &= t_1\sqrt{\left(\frac{5.2r_1}{r_1}\right)^3} \\ &= 1\,\text{year} \times \sqrt{5.2^3} \\ &= 12\,\text{years} \end{aligned}$$

QUESTION 6.24

(i) Putting $D = 2r$ in Equation 6.38, $A_{sph} = 4\pi(D/2)^2 = 4\pi D^2/4 = \pi D^2$.

(ii) With $D = 2r$ in Equation 6.39, $V_{sph} = 4\pi(D/2)^3/3 = 4\pi D^3/(3 \times 8) = \pi D^3/6$.

QUESTION 6.25

From Equation 6.35, $C = 2\pi \times 1.50 \times 10^{11}\,\text{m} = 9.42 \times 10^{11}\,\text{m}$.

QUESTION 6.26

(i) Using Equation 6.38, $A = 4\pi \times (6.96 \times 10^8\,\text{m})^2 = 6.09 \times 10^{18}\,\text{m}^2$.

(ii) Using Equation 6.39, $V = 4\pi \times (6.96 \times 10^8\,\text{m})^3/3 = 1.41 \times 10^{27}\,\text{m}^3$.

QUESTION 6.27

60° is one-sixth of a complete turn, so from Equation 6.41, $60° = \pi/3$ radians. From Equation 6.43, we have $60 \times (2\pi/360)$ radians = 1.047 radians.

QUESTION 6.28

From Equation 6.41, we know $\pi/4$ radians must correspond to one-eighth of a complete turn, that is 45°. Alternatively, from Equation 6.42, we have $(360°/2\pi) \times \pi/4 = 45°$.

QUESTION 6.29

1 arcsec = 1 arcmin/60 = 1°/3600. So 1 arcsec = (2π/360) × (1/3600) radians = 4.847×10^{-6} radians.

QUESTION 6.30

Suppose the unknown side is y. Then from Equation 6.44, $y^2 = h^2 - x^2$ $= (7.2\,\text{cm})^2 - (6.5\,\text{cm})^2 = 51.84\,\text{cm}^2 - 42.25\,\text{cm}^2 = 9.59\,\text{cm}^2$ so

$$y = \sqrt{9.59\,\text{cm}^2} = 3.1\,\text{cm}$$

QUESTION 6.31

The distance is the hypotenuse, h, of the right-angled triangle in Figure 6.7. We know the two other sides so we can use Equation 6.46:

$$x/h = \cos\theta$$

so $x = h\cos\theta$

$$h = x/\cos\theta = 20\,\text{m}/\cos 60° = 40\,\text{m}$$

Or use Equation 6.45 and the answer from Example 6.17:

$$y/h = \sin\theta$$

$$h = y/\sin\theta = 35\,\text{m}/\sin 60° = 40\,\text{m}$$

QUESTION 6.32

Use the same method as in Question 6.31, but switch your calculator into radian mode. Either use Equation 6.46:

$$x/h = \cos\theta$$

so $x = h\cos\theta$

$$h = x/\cos\theta = 20\,\text{m}/\cos(\pi/3\,\text{rad}) = 40\,\text{m}$$

Or use Equation 6.45 and the answer from Example 6.17:

$$y/h = \sin\theta$$

$$h = y/\sin\theta = 35\,\text{m}/\sin(\pi/3\,\text{rad}) = 40\,\text{m}.$$

QUESTION 6.33

From Equation 6.49 and Example 6.20, $\theta = 8.7 \times 10^{-3}$ rad and distance $r = d/\alpha$ = $3.5 \times 10^6\,\text{m}/8.7 \times 10^{-3}\,\text{rad} = 4 \times 10^8\,\text{m}$.

QUESTION 6.34

If we choose the points $t_1 = 0.0\,\text{s}$, $v_1 = 3.0\,\text{m}\,\text{s}^{-1}$, and $t_2 = 5.0\,\text{s}$, $v_2 = 13.0\,\text{m}\,\text{s}^{-1}$, we have from Equation 6.55:

$$\text{gradient} = (13.0 - 3.0)\,\text{m}\,\text{s}^{-1}/(5.0 - 0.0)\,\text{s} = 10.0\,\text{m}\,\text{s}^{-1}/5.0\,\text{s} = 2.0\,\text{m}\,\text{s}^{-2}$$

The gradient here is equal to the object's *acceleration*: it is found by dividing a change in speed by the time interval over which it occurred.

QUESTION 6.35

From Figure 6.25, rise = $(13.6 - 9.0) \times 10^5$ km = 4.6×10^5 km,
run = $(5.50 - 2.15)$ yr = 3.35 yr.

From Equation 6.55, gradient = 4.6×10^5 km/3.35 yr ≈ 1.4×10^5 km yr^{-1}. (Any value close to 1.4×10^5 km yr^{-1} is fine, because your exact value will depend on where you drew the line.)

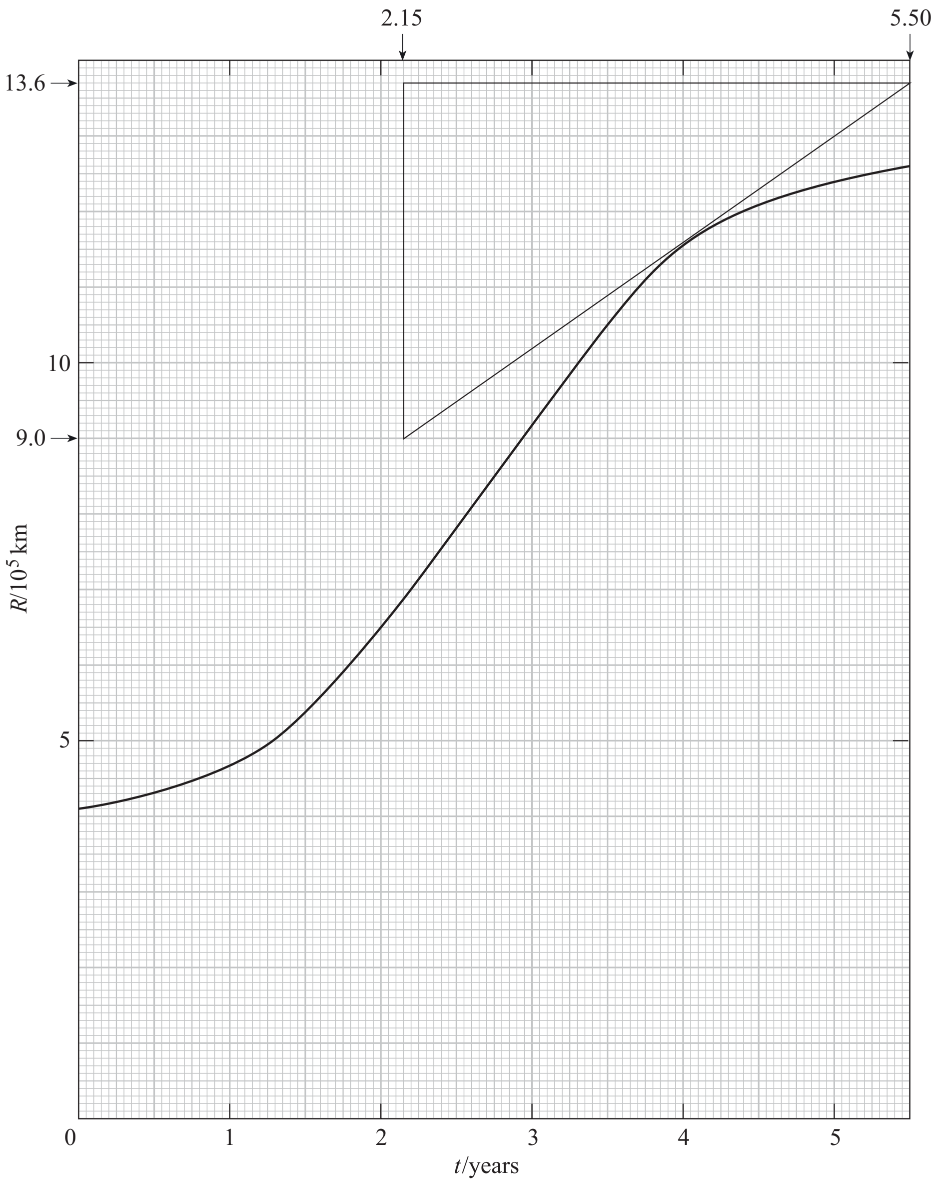

Figure 6.25 Graph for the answer to Question 6.35.

QUESTION 6.36

See Figure 6.26. Table 6.5 lists the logarithms for plotting the graph in Figure 6.26.

Table 6.5 Logarithms of the values listed in Table 6.4.

Planet	$\log_{10}(d/10^6\,\mathrm{km})$	$\log_{10}(T/\mathrm{yr})$
Mercury	1.76	–0.620
Venus	2.04	–0.208
Earth	2.18	0.00
Mars	2.36	0.279
Jupiter	2.89	1.08
Saturn	3.15	1.46
Uranus	3.46	1.92
Neptune	3.65	2.20
Pluto	3.77	2.40

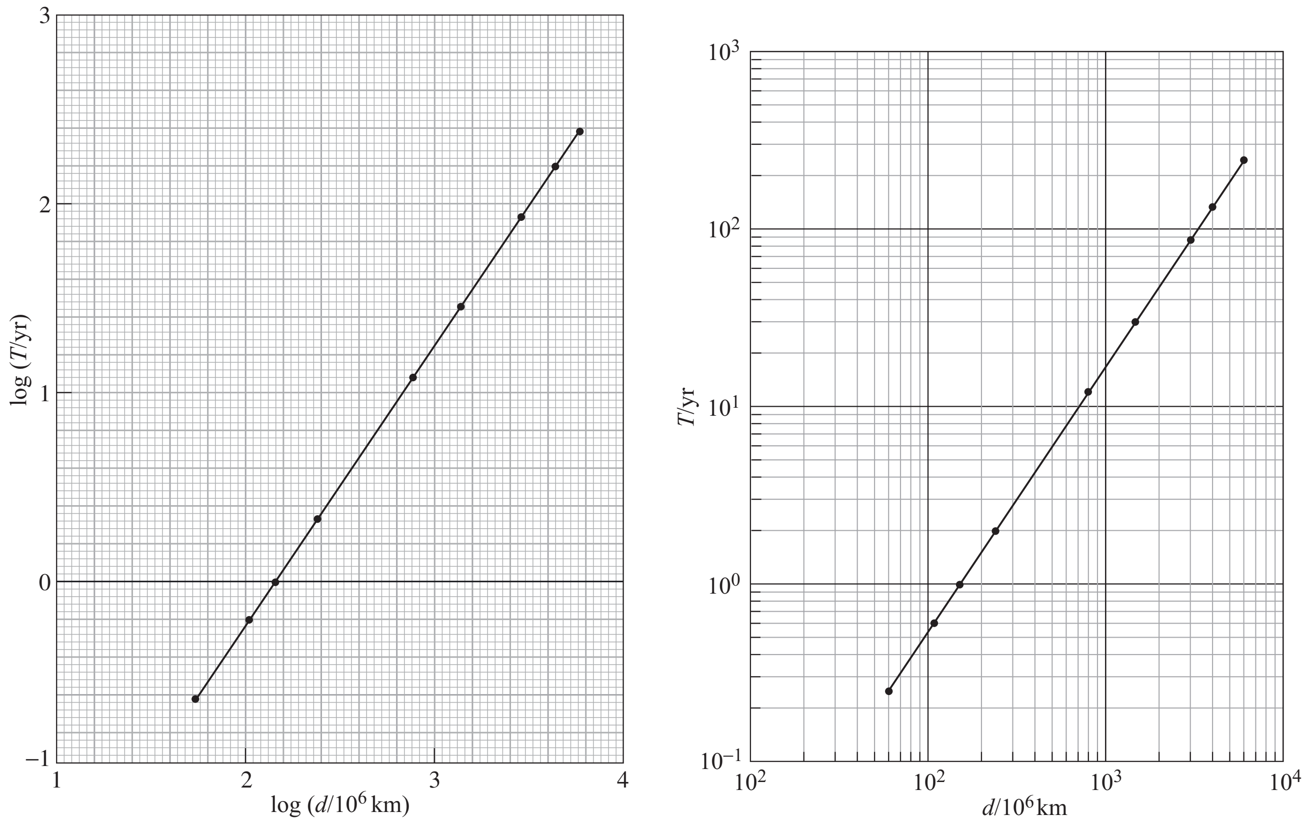

Figure 6.26 The answer to Question 6.36: (left) logs of the values plotted on ordinary graph paper (right) values plotted on logarithmic graph paper

QUESTION 6.37

$\log_{10}(h/\text{km}) = \log_{10}(5 \times 10^3) = 3.7$. From the graph, the temperature at this height has $\log(T/\text{K}) = 5.6$, so $T/\text{K} = \text{antilog}_{10}(5.6) \approx 4.0 \times 10^5$, therefore $T \approx 4.0 \times 10^5$ K.

QUESTION 6.38

The point for 3×10^4 km lies about half-way along from 1×10^4 km. From Figure 6.27, the point for the corresponding temperature is about one-fifth of the way up from 1×10^6 K. so the temperature is about 1.5×10^6 K.

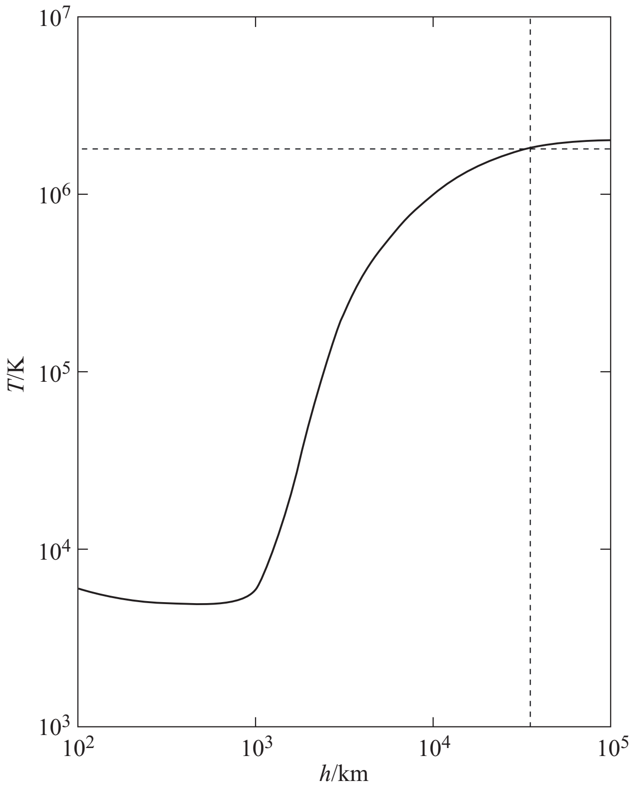

Figure 6.27 Reading the graph to answer Question 6.38.